Flimsy Spaces

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Definition. Let $n \geq 1$. A topological space X is said to be n-flimsy if removing fewer then n arbitrary points leaves the space connected and removing any n arbitrary (distinct) points disconnects the space.

For example, $\mathbb{R}$ is 1-flimsy and S^1 is 2-flimsy. In the following, we prove 3-flimsy spaces does not exist.

Theorem 1. *Let X be a 2-flimsy space and $x, y \in X$, with $x \neq y$.*

If there is three open sets of X, U_1, U_2, and U_3, such that $(U_1 \cup U_2 \cup U_3) \cap \{x, y\}^c = X \backslash \{x, y\}$ and $U_1 \cap U_2 \cap \{x, y\}^c = U_1 \cap U_3 \cap \{x, y\}^c = U_2 \cap U_2 \cap \{x, y\}^c = \emptyset$, then there is $i \in \{1, 2, 3\}$ such that $U_i \cap \{x, y\}^c = \emptyset$

So, $X \backslash \{x, y\}$ has exactly two connected components.

Proof. We prove it by contradiction: let us suppose $\forall i \in \{1, 2, 3\},\ U_i \cap \{x, y\}^c \neq \emptyset$. We choose $u_1 \in U_1 \cap \{x, y\}^c$ and $u_2 \in U_2 \cap \{x, y\}^c$. $u_1 \notin U_2 \cup U_3$ and $u_2 \notin U_1 \cup U_3$. We are going to prove $X \backslash \{u_1, u_2\}$ is connected, which contradicts that X is 2-flimsy. Let U, V two open sets of X such that $(U \cup V) \cap \{u_1, u_2\}^c = X \backslash \{u_1, u_2\}$ and $U \cap V \cap \{u_1, u_2\}^c = \emptyset$. We can suppose $x \in U$ without loss of generality, and so $x \notin V$

1. $U \cup U_1 \cup U_2$ and $V \cap U_3$ are open.

   $(U \cup U_1 \cup U_2) \cap (V \cap U_3) \subset (U \cap V) \cup (U_1 \cap U_3) \cup (U_2 \cap U_3) \subset \{u_1, u_2, x, y\}$ but $x \notin V$, and $u_1, u_2 \notin U_3$ so $(U \cup U_1 \cup U_2) \cap (V \cap U_3) \cap \{y\}^c = \emptyset$

   $(U \cup U_1 \cup U_2) \cup (V \cap U_3) \supset U_1 \cup U_2 \cup (U_3 \cap (U \cup V)) \supset (U_1 \cup U_2 \cup U_3) \cap \{u_1, u_2\}^c \supset X \backslash \{u_1, u_2, x, y\}$ but $x \in U$, $u_1 \in U_1$, and $u_2 \in U_2$ so $((U \cup U_1 \cup U_2) \cup (V \cap U_3)) \cap \{y\}^c = X \backslash \{y\}$

   X is 2-flimsy so $X \backslash \{y\}$ is connected. Moreover $x \in (U \cup U_1 \cup U_2) \cap \{y\}^c \neq \emptyset$.

   So $(V \cap U_3) \cap \{y\}^c = \emptyset$

2. If $y \in V$, then $y \notin U$ and the previous step implies $(U \cap U_3) \cap \{x\}^c = \emptyset$. Then $U_3 \cap \{x, y\}^c \subset (U_3 \cap U \cap \{x\}^c) \cup (U_3 \cap V \cap \{y\}^c) \cup (U_3 \cap \{u_1, u_2\}) = \emptyset$ which is false.

   So $y \in U$, $y \notin V$, $V \cap U_3 = \emptyset$, and $U_3 \subset U$

3. $U \cup U_1$ and $V \cap U_2$ are open.

   $(U \cup U_1) \cap (V \cap U_2) \subset (U \cap V) \cup (U_1 \cap U_2) \subset \{x, y, u_1, u_2\}$ but $u_1 \notin U_2$ and $x, y \notin V$ so $(U \cup U_1) \cap (V \cap U_2) \cap \{u_2\}^c = \emptyset$

   $(U \cup U_1) \cup (V \cap U_2) \supset U_1 \cup U \cup (U_2 \cap (U \cup V)) \supset (U_1 \cup U_3 \cup U_2) \cap \{u_1, u_2\}^c \supset X \backslash \{u_1, u_2, x, y\}$ but $x, y \in U$, and $u_1 \in U_1$ so $((U \cup U_1) \cup (V \cap U_2)) \cap \{u_2\}^c = X \backslash \{u_2\}$

   $X \backslash \{u_2\}$ is connected and $x \in (U \cup U_1) \cap \{u_2\}^c \neq \emptyset$ so $(V \cap U_2) \cap \{u_2\}^c = \emptyset$

4. With the same previous step, we have $(V \cap U_1) \cap \{u_1\}^c = \emptyset$.

   So $V \cap \{u_1, u_2\}^c \subset (V \cap U_1 \cap \{u_1\}^c) \cup (V \cap U_2 \cap \{u_2\}^c) \cup (V \cap (U_3 \cup \{x, y\})) = \emptyset$. So, $X \backslash \{u_1, u_2\}$ is connected.

□

Theorem 2. *A n-flimsy space is infinite.*

Proof. see https://math.stackexchange.com/questions/2939445/flimsy-spaces-removing-any-n-points-results-in-disconnectedness for the proof of 'Babelfish' □

Theorem 3. *Let X a n-flimsy space. $\forall x \in X$, $\{x\}$ is either open or closed.*

Proof. We start with the case $n = 1$. X is connected but $X \backslash \{x\}$ is disconnected. It exists a nontrivial clopen set $Y \subset X \backslash \{x\}$, in particular $Y \neq \emptyset$ and $Y \cup \{x\} \neq X$. Since Y is open in $X \backslash \{x\}$, Y or $Y \cup \{x\}$ is open in X.

- if Y is open in X, by connectedness, Y is not closed in X. Since Y in closed in $X \backslash \{x\}$, $Y \cup \{x\}$ is closed in X. So, $\{x\} = (Y \cup \{x\}) \cap (X \backslash Y)$ is closed.
- if $Y \cup \{x\}$ is open in X, then Y is closed in X, and $\{x\} = (Y \cup \{x\}) \cap (X \backslash Y)$ is open.

By induction, we suppose the theorem to be true for $n \geq 1$, and we observe X a $(n+1)$-flimsy space and $x \in X$. X is infinite, so there is $y, z \in X$, $y \neq z$, such that $\{x\}$ is either open in $X \backslash \{y\}$ and $X \backslash \{z\}$ or closed in $X \backslash \{y\}$ and $X \backslash \{z\}$, because $X \backslash \{y\}$ and $X \backslash \{z\}$ are n-flimsy. We suppose we are in the open case (the closed space can be examined in the same way).

If $\{x\}$ is not open in X then $\{x, y\}$ and $\{x, z\}$ are open in X, so $\{x\} = \{x, y\} \cap \{x, z\}$ is open in X.

□

Lemma 1. *Let $x, t, s \in X$, three distinct points of a 2-flimsy space. We denote $C_1(t), C_2(t)$ the two connected components of $X \backslash \{x, t\}$ and $C_1(s), C_2(s)$ the two connected components of $X \backslash \{x, s\}$. We suppose $s \in C_1(t)$ and $t \in C_1(s)$.*

$D = C_1(t) \cap C_1(s)$ is one of the two connected components of $X \backslash \{t, s\}$

Proof. We begin by showing $C_1(t) \cup \{x\}$ is connected by contradiction: we suppose it is disconnected.

$C_1(t)$ is open and closed in $X\backslash\{x,t\}$, because it is one of the only two connected components. Moreover, $C_1(t) \cup \{x\}$ has also two connected components, $C_1(t)$ and $\{x\}$, so $C_1(t)$ is open and closed in $C_1(t) \cup \{x\} \subset X\backslash\{t\}$.

So $C_1(t)$ or $C_1(t) \cup \{x\}$ is open in $X\backslash\{t\}$, but we know there is an open set U of $X\backslash\{t\}$ such that $C_1(t) = U \cap (C_1(t) \cup \{x\})$, so in every case, $C_1(t)$ is open in $X\backslash\{t\}$. The same shows $C_1(t)$ is closed in $X\backslash\{t\}$. $C_1(t)$ is not trivial so $X\backslash\{t\}$ is not connected: we have a contradiction.

Of course, $C_i(r) \cup \{y\}$ is connected, for $i = 1$ or 2, $r = t$ or s, and $y = x$ or r.

$X\backslash\{t,s\} = D \cup (C_2(t) \cup \{x\}) \cup (C_2(s) \cup \{x\})$, and $(C_2(t) \cup \{x\}) \cup (C_2(s) \cup \{x\})$ is connected. We only need to show D is connected.

If D is not connected, there are U, V open sets of X such that $U \cap V \cap D = \emptyset$, $(U \cup V) \cap D = D$, and $U \cap D \neq \emptyset$ and $V \cap D \neq \emptyset$. Let $u \in U \cap D$ and $v \in V \cap D$. $X\backslash\{u,v\}$ is not connected, we have $\tilde{U}, \tilde{V}$ open sets of X such that $\tilde{U} \cap \tilde{V} \cap \{u,v\}^c = \emptyset$, $(\tilde{U} \cup \tilde{V}) \cap \{u,v\}^c = X\backslash\{u,v\}$, and $\tilde{U} \cap \{u,v\}^c \neq \emptyset$ and $\tilde{V} \cap \{u,v\}^c \neq \emptyset$.

By connectedness of $(C_2(t) \cup \{x\}) \cup (C_2(s) \cup \{x\}) = D^c \cap \{u,v\}^c = D^c$, we can suppose $D^c \subset \tilde{U}$ and $\tilde{V} \subset D$

$(V \cap \tilde{V} \cap \{u,v\}^c) \cup (U \cap \tilde{V} \cap \{u,v\}^c) = \tilde{V} \cap \{u,v\}^c \cap (U \cup V) = \tilde{V} \cap \{u,v\}^c \cap D \cap (U \cup V) = \tilde{V} \cap \{u,v\} \cap D = \tilde{V} \cap \{u,v\}^c \neq \emptyset$ so we can suppose $V \cap \tilde{V} \cap \{u,v\}^c \neq \emptyset$

$U \cup \tilde{U}$ and $V \cap \tilde{V}$ are open.

$(U \cup \tilde{U}) \cap (V \cap \tilde{V}) \subset (U \cap V \cap \tilde{V}) \cup (\tilde{U} \cap \tilde{V}) \subset (U \cap V \cap D) \cup \{u,v\} = \{u,v\}$ but $u \notin V$, so $(U \cup \tilde{U}) \cap (V \cap \tilde{V}) \cap \{v\}^c = \emptyset$

$(U \cup \tilde{U}) \cup (V \cap \tilde{V}) \supset \tilde{U} \cup (\tilde{V} \cap (U \cup V)) \supset \tilde{U} \cup (\tilde{V} \cap D) = \tilde{U} \cap \tilde{V} \supset X\backslash\{u,v\}$ but $u \in U$ so $((U \cup \tilde{U}) \cup (V \cap \tilde{V})) \cap \{v\}^c = X\backslash\{v\}$

Moreover, $u \in (U \cup \tilde{U}) \cap \{v\}^c \neq \emptyset$ and $(V \cap \tilde{V}) \cap \{v\}^c \supset V \cap \tilde{V} \cap \{u,v\}^c \neq \emptyset$ so $X\backslash\{v\}$ is not connected: contradiction.

We have proven D is connected.

□

Theorem 4. *There are no* 3*-flimsy spaces.*

Proof. Let X a 3-flimsy space and x, y, t, s some distinct points of X. $X\backslash\{y\}$ is 2-flimsy, so if $C_1(t)$ is the connected component of $X\backslash\{y,x,t\}$ containing s and $C_1(s)$ is the connected component of $X\backslash\{y,x,s\}$ containing t, then $D = C_1(t) \cap C_1(s)$ is one of the two connected components of $X\backslash\{y,t,s\}$. Moreover, D is also one of the two connected components of $X\backslash\{x,t,s\}$. $x, y, t, s \notin D$

So, D is open and closed in $X\backslash\{x,t,s\}$ and in $X\backslash\{y,t,s\}$. We have two open sets of X, U_x and U_y, and two closed sets of X, G_x and G_y, such that

$U_x \cap \{x,t,s\}^c = G_x \cap \{x,t,s\}^c = D$ and $U_y \cap \{y,t,s\}^c = G_y \cap \{y,t,s\}^c = D$, so $y \notin U_x, G_x$ and $x \notin U_y, G_y$

$U_x \cap U_y \cap \{t,s\}^c = U_x \cap \{y,t,s\}^c \cap U_y \cap \{x,t,s\}^c = D \cap D = D$ and also, $G_x \cap G_y \cap \{t,s\}^c = D$. Since $U_x \cap U_y$ is open in X and $G_x \cap G_y$ is closed in X, D is open and closed in $X \backslash \{t,s\}$. Moreover, D is not trivial because it is a connected component of $X \backslash \{x,t,s\}$. So $X \backslash \{t,s\}$ is not connected and X is not 3-flimsy.

□