An algebraic lemma

Béranger Seguin

May 28, 2022

Definition 0.1. A subset of a ring is *(non-)nilpotent* if it is (not) included in $\sqrt{0}$. An injective morphism of rings $f : A \to B$ is *faithful* if for any non-nilpotent ideal I of B, $f^{-1}(I)$ is non-nilpotent.

Lemma 0.2. *Let $f : A \to B$ be a faithful injection. Assume B is Noetherian. Let p be a minimal prime ideal of B and $q = f^{-1}(p)$. Then the induced injection $f_{/p} : A/q \to B/p$ is faithful.*

Proof.

- Let V be the intersection of all minimal prime ideals of B distinct from p. The set $p \cap V$ is the intersection of all minimal prime ideals of B, and is therefore equal to $\sqrt{0}$.

- Let O be an ideal such that $O \cap V$ is nilpotent. Let us show that O is included in p.

  First observe that V is not included in p: indeed, the ideal p is prime and V is a finite intersection of ideals (by Noetherianity). If V were included in p, some minimal prime ideal distinct from p would be included in p, which contradicts the minimality of p. Hence we can choose an element y which is in V but not in p.

  Now consider an arbitrary element $o \in O$. We have $oy \in O \cap V \subseteq \sqrt{0} \subseteq p$. Since p is prime and y is not in p, this implies that $o \in p$. Hence $O \subseteq p$.

- Consider a nonzero ideal I of B/p.[1] Lift it into an ideal I' of B not included in p. Since I' is not included in p, the previous point shows that $I' \cap V$ is non-nilpotent. By faithfulness of f, $f^{-1}(I' \cap V)$ is non-nilpotent.

  To prove that $f_{/p}^{-1}(I)$ is non-nilpotent (i.e. nonzero), one only needs to show that $f^{-1}(I')$ is not included in q. But if $f^{-1}(I')$ were included in q, then we would have:

$$f^{-1}(I' \cap V) \subseteq q \cap f^{-1}(V) = f^{-1}(p \cap V) = f^{-1}(\sqrt{0}) \overset{f \text{ injective}}{=} \sqrt{0}.$$

  This contradicts the fact that $f^{-1}(I' \cap V)$ is non-nilpotent.

$\square$

[1] Since B/p and A/q are integral, nonzero and non-nilpotent ideals are the same thing.