Rings of Components of Hurwitz Spaces

Béranger Seguin

Laboratoire Paul Painlevé, Université de Lille

May 5, 2022

Outline

- I. Motivation, context, and main objects:
  - The Inverse Galois Problem
  - G-covers
  - Hurwitz spaces
  - Connected components of Hurwitz spaces
- II. The results of Ellenberg-Venkatesh-Westerland:
  - The ring of components
  - A description of their approach
  - The non-splitting property
- III. What I've been working on:
  - The geometry of the ring of components
  - The example of the symmetric group
  - The ring of components and arithmetical questions

The Inverse Galois Problem

Let G be a finite group. We may ask various questions:

Question (Inverse Galois Problem, IGP)

Is G the Galois group of a finite Galois extension of $\mathbb{Q}$?

$$\begin{array}{c} L \\ G \,\big| \\ \mathbb{Q} \end{array}$$

Or: is G the Galois group of a polynomial $P \in \mathbb{Q}[X]$?

The Inverse Galois Problem

Let G be a finite group. We may ask various questions:

Question (Inverse Galois Problem, IGP)

Is G the Galois group of a finite Galois extension of $\mathbb{Q}$?

Question (Regular Inverse Galois Problem, RIGP)

Is G the Galois group of a regular finite extension of $\mathbb{Q}(t)$?

$$\begin{array}{c} L \\ G \Big| \\ \mathbb{Q}(t) \end{array} \quad \text{with } L \cap \overline{\mathbb{Q}} = \mathbb{Q}$$

Hilbert's Irreducibility Theorem $\rightsquigarrow$ RIGP implies IGP.

The Inverse Galois Problem

Let G be a finite group. We may ask various questions:

Question (Inverse Galois Problem, IGP)

Is G the Galois group of a finite Galois extension of $\mathbb{Q}$?

Question (Regular Inverse Galois Problem, RIGP)

Is G the Galois group of a regular finite extension of $\mathbb{Q}(t)$?

Question (Geometric reformulation of RIGP)

Is there a connected branched G-cover of the complex projective line defined over $\mathbb{Q}$?

The Inverse Galois Problem

Let G be a finite group. We may ask various questions:

Question (Inverse Galois Problem, IGP)

Is G the Galois group of a finite Galois extension of $\mathbb{Q}$?

Question (Regular Inverse Galois Problem, RIGP)

Is G the Galois group of a regular finite extension of $\mathbb{Q}(t)$?

Question (Geometric reformulation of RIGP)

Is there a connected branched G-cover of the complex projective line defined over $\mathbb{Q}$?

Question

Does the "Hurwitz space" $\mathrm{CHur}(G, n)$ admit a $\mathbb{Q}$-rational point, for some $n \in \mathbb{N}$?

$\mathrm{CHur}(G, n)$ = space of connected G-covers with n branch points (cf. later)

The configuration space

Definition

Let $n \in \mathbb{N}$.

- **A configuration :** an unordered list of n distinct complex numbers;
- **The space** Conf_n **:** the space of configurations;

The configuration space

Definition

Let $n \in \mathbb{N}$.

- **A configuration :** an unordered list of n distinct complex numbers;
- **The space** Conf_n **:** the space of configurations;

$$\mathrm{Conf}_n = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; i \neq j \Rightarrow z_i \neq z_j \right\} / \mathfrak{S}_n$$

The configuration space

Definition

Let $n \in \mathbb{N}$.

- **A configuration :** an unordered list of n distinct complex numbers;
- **The space** Conf_n **:** the space of configurations;

$$\begin{aligned} \mathrm{Conf}_n &\simeq \{\text{separable unitary polynomials} \in \mathbb{C}[X] \text{ of degree } n\} \\ &\simeq \mathbb{P}^n(\mathbb{C}) \setminus \Delta \end{aligned}$$

where Δ = big diagonal = zero locus of the discriminant map (polynomial condition)

The configuration space

Definition

Let $n \in \mathbb{N}$.

- **A configuration :** an unordered list of n distinct complex numbers;
- **The space** Conf_n **:** the space of configurations;
- **The braid group** B_n **:** its fundamental group $\pi_1(\mathrm{Conf}_n)$.

Generators and relations:

$$\mathrm{B}_n \simeq \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ccll} \sigma_i \sigma_j & = & \sigma_j \sigma_i & \text{when } |i-j| \geq 2 \\ \sigma_i \sigma_{i+1} \sigma_i & = & \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{if } i \in \{1, \ldots, n-2\} \\ \sigma_1 \ldots \sigma_{n-1} \sigma_{n-1} \ldots \sigma_1 & = & 1 & \end{array} \right\rangle.$$

G-covers

Definition (Branched G-cover)

A G-cover of $\mathbb{P}^1(\mathbb{C})$ branched at $\underline{t} = (t_1, \ldots, t_n) \in \mathrm{Conf}_n$ is:

- a covering map $p : Y \to \mathbb{P}^1(\mathbb{C}) \setminus \{t_1, \ldots, t_n\}$;
- a morphism $\alpha : G \to \mathrm{Aut}(p)$ which induces a free transitive action on fibers.

Marked G-covers come with an additional marked point $\star \in p^{-1}(\infty)$.

G-covers

Definition (Branched G-cover)

A G-cover of $\mathbb{P}^1(\mathbb{C})$ branched at $\underline{t} = (t_1, \ldots, t_n) \in \mathrm{Conf}_n$ is:

- a covering map $p : Y \to \mathbb{P}^1(\mathbb{C}) \setminus \{t_1, \ldots, t_n\}$;
- a morphism $\alpha : G \to \mathrm{Aut}(p)$ which induces a free transitive action on fibers.

Marked G-covers come with an additional marked point $\star \in p^{-1}(\infty)$.

Other ways to define covers:

- As the zero locus of a polynomial in two variables;
- As a dominant morphism from a smooth curve to $\mathbb{P}^1_{\mathbb{C}}$;
- As a field extension of $\mathbb{C}(t)$.

Hurwitz spaces

Definition (Hurwitz space)

The Hurwitz space $\mathrm{Hur}(G, n)$:

- Topological space
- Points = isom classes of marked G-covers of $\mathbb{P}^1(\mathbb{C})$ with n branch points

Hurwitz spaces

Definition (Hurwitz space)

The Hurwitz space $\mathrm{Hur}(G, n)$:

- Topological space
- Points = isom classes of marked G-covers of $\mathbb{P}^1(\mathbb{C})$ with n branch points

$$\begin{array}{c}\mathrm{Hur}(G, n) \\ \downarrow \\ \mathrm{Conf}_n\end{array}$$

- Itself a covering space of Conf_n: cover $\mapsto$ branch points;

Hurwitz spaces

Definition (Hurwitz space)

The Hurwitz space $\mathrm{Hur}(G, n)$:

- Topological space
- Points = isom classes of marked G-covers of $\mathbb{P}^1(\mathbb{C})$ with n branch points

$$\begin{array}{c} \mathrm{Hur}(G, n) \\ \downarrow \\ \mathrm{Conf}_n \end{array}$$

- Itself a covering space of Conf_n: cover $\mapsto$ branch points;
- The fiber above $\underline{t} \in \mathrm{Conf}_n$:

$$\left\{ \begin{array}{c} \text{Isomorphism classes of marked} \\ G\text{-covers branched at } \underline{t} \end{array} \right\}$$

Hurwitz spaces

Definition (Hurwitz space)

The Hurwitz space $\mathrm{Hur}(G, n)$:

- Topological space
- Points = isom classes of marked G-covers of $\mathbb{P}^1(\mathbb{C})$ with n branch points

$$\begin{array}{c} \mathrm{Hur}(G,n) \\ \downarrow \\ \mathrm{Conf}_n \end{array}$$

- Itself a covering space of Conf_n: cover $\mapsto$ branch points;
- The fiber above $\underline{t} \in \mathrm{Conf}_n$:

$$\left\{ \begin{array}{c} \text{Isomorphism classes of marked} \\ G\text{-covers branched at } \underline{t} \end{array} \right\}$$

- Lifting a path in Conf_n
  $\Leftrightarrow$ deform covers when branch points move around

G-covers as morphisms

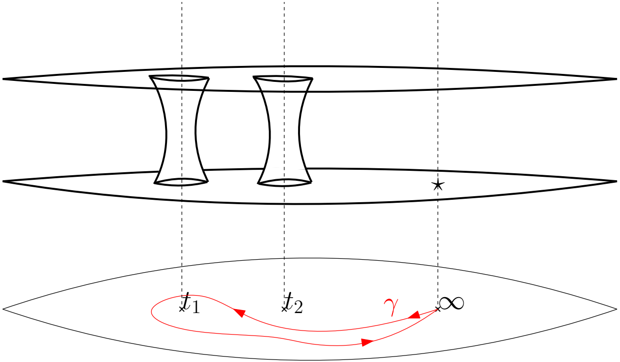
G-covers : may be seen as group morphisms from a fundamental group to G

$$\varphi : \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \underline{t}, \infty) \to G$$

G-covers as morphisms: explanation

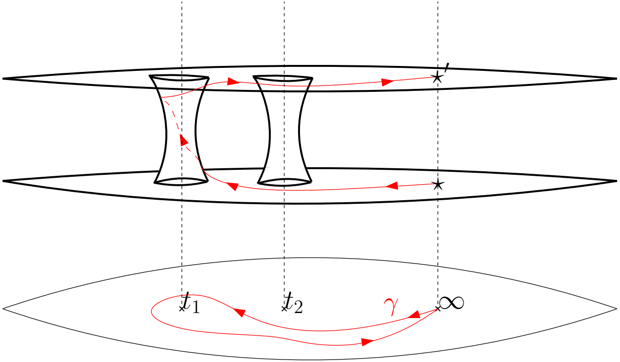
- Consider:

$$\begin{aligned} p &= \text{a marked } G\text{-cover branched at } \underline{t} \\ \gamma &= \text{a loop } \in \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \underline{t}, \infty) \end{aligned}$$



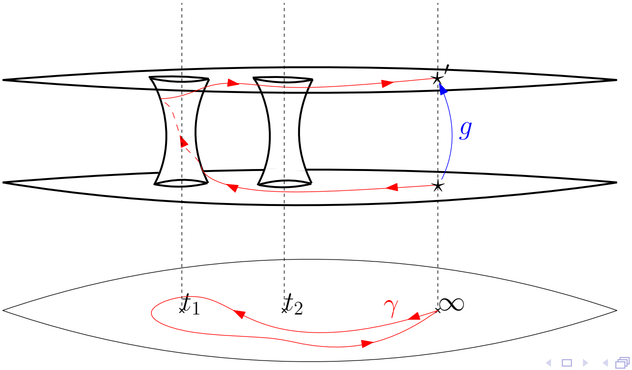
G-covers as morphisms: explanation

- Consider: p = a marked G-cover branched at $\underline{t}$
  γ = a loop $\in \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \underline{t}, \infty)$
- There is a unique path starting at $\star$ that lifts γ $\rightsquigarrow$ its endpoint is some $\star' \in p^{-1}(\infty)$



G-covers as morphisms: explanation

- Consider: p = a marked G-cover branched at $\underline{t}$
  γ = a loop $\in \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \underline{t}, \infty)$
- There is a unique path starting at $\star$ that lifts γ $\rightsquigarrow$ its endpoint is some $\star' \in p^{-1}(\infty)$
- G acts freely/transitively on $p^{-1}(\infty)$ $\rightsquigarrow$ $\star' = g.\star$ for a unique $g \in G$.



G-covers as morphisms: explanation

- The map $\gamma \mapsto g$ is the *monodromy morphism*:

$$\varphi : \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \underline{t}, \infty) \to G;$$

G-covers as morphisms: explanation

- The map $\gamma \mapsto g$ is the *monodromy morphism*:

$$\varphi : \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \underline{t}, \infty) \to G;$$

- The morphism φ characterizes p up to iso:

$$\left\{ \begin{array}{c} \text{Isomorphism classes of marked} \\ G\text{-covers branched at } \underline{t} \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{group morphisms} \\ \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \underline{t}, \infty) \to G \end{array} \right\}$$

G-covers as morphisms: explanation

- The map $\gamma \mapsto g$ is the *monodromy morphism*:

$$\varphi : \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \underline{t}, \infty) \to G;$$

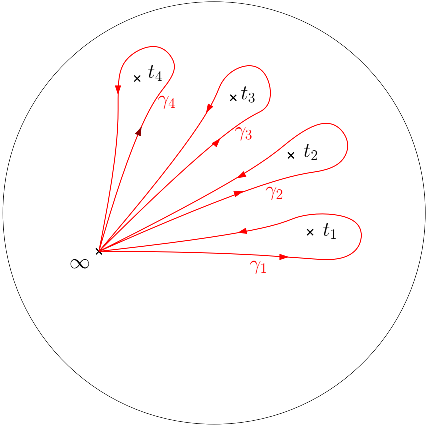
- The morphism φ characterizes p up to iso:

$$\left\{ \begin{matrix} \text{Isomorphism classes of marked} \\ G\text{-covers branched at } \underline{t} \end{matrix} \right\} \simeq \left\{ \begin{matrix} \text{group morphisms} \\ \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \underline{t}, \infty) \to G \end{matrix} \right\}$$

- Connected cover $\Leftrightarrow$ Surjective morphism
  Non-marked cover $\Leftrightarrow$ Morphism modulo $\mathrm{Inn}(G)$

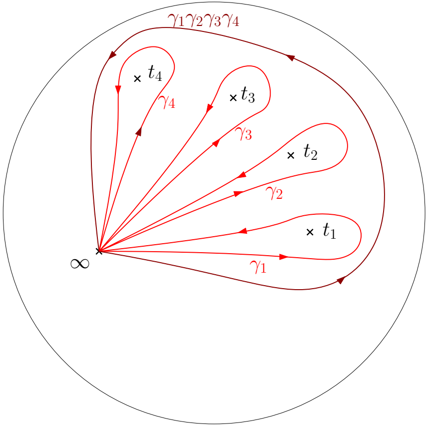
G-covers as tuples

- The group $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \underline{t}, \infty)$ is non-canonically isomorphic to the group:
$$\langle \gamma_1, \ldots, \gamma_n \,|\, \gamma_1 \ldots \gamma_n = 1 \rangle \,.$$
- An isomorphism is given by any choice of a *topological bouquet*:



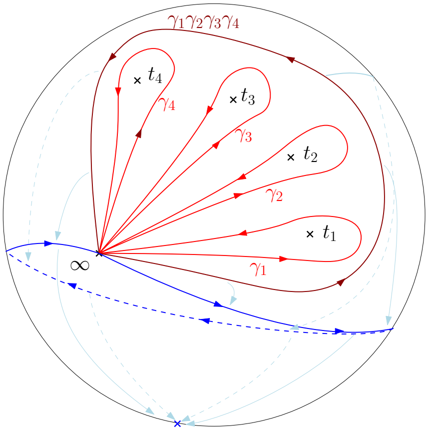
G-covers as tuples

- The group $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \underline{t}, \infty)$ is non-canonically isomorphic to the group:
$$\langle \gamma_1, \dots, \gamma_n \,|\, \gamma_1 \dots \gamma_n = 1 \rangle \,.$$
- Can you see why the relation $\gamma_1\gamma_2\gamma_3\gamma_4 = 1$ holds?



G-covers as tuples

- The group $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \underline{t}, \infty)$ is non-canonically isomorphic to the group:
$$\langle \gamma_1, \ldots, \gamma_n \mid \gamma_1 \ldots \gamma_n = 1 \rangle .$$
- I did my best at illustrating it...



G-covers as tuples

- The group $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \underline{t}, \infty)$ is non-canonically isomorphic to the group:
$$\langle \gamma_1, \ldots, \gamma_n \mid \gamma_1 \ldots \gamma_n = 1 \rangle .$$
- Covers correspond to morphisms:
$$\varphi : \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \underline{t}, \infty) \to G$$
and thus to tuples:
$$\underline{g} = (g_1, \ldots, g_n) \in G^n \text{ with } g_1 \ldots g_n = 1.$$

(let $g_i = \varphi(\gamma_i)$)

G-covers as tuples: the punchline

$\rightsquigarrow$ **combinatorial description of G-covers as tuples**

Iso class of marked G-covers branched at $\underline{t} \in \mathrm{Conf}_n$	$\Leftrightarrow$	Tuple $(g_1, \ldots, g_n) \in G^n$ such that $g_1 \cdot g_2 \cdot \ldots \cdot g_n = 1$
Connected cover	$\Leftrightarrow$	$g_1, \ldots, g_n$ generate G
Non-marked cover	$\Leftrightarrow$	Tuple considered modulo $\mathrm{Inn}(G)$

Connected components of the Hurwitz space

- Recall that we look for $\mathbb{Q}$-rational points of Hur.
  $\rightsquigarrow$ We need to find connected components defined over $\mathbb{Q}$.
  $\rightsquigarrow$ a problem on the way to (R)IGP! $\rightsquigarrow$ describe the connected components?

Connected components of the Hurwitz space

- Recall that we look for $\mathbb{Q}$-rational points of Hur.
  $\rightsquigarrow$ We need to find connected components defined over $\mathbb{Q}$.
  $\rightsquigarrow$ a problem on the way to (R)IGP! $\rightsquigarrow$ describe the connected components?
- Combinatorial description of connected components of $\mathrm{Hur}(G, n)$:

Proposition (Combinatorial description of the connected components of $\mathrm{Hur}(G, n)$)

$$\left\{ \begin{matrix} \textit{Connected components} \\ \textit{of } \mathrm{Hur}(G, n) \end{matrix} \right\} \underset{\textit{non-canonical}}{\simeq} \left\{ \begin{matrix} n-\textit{tuples } (g_1, \ldots, g_n) \in G^n \\ \textit{such that } g_1 \cdot g_2 \cdots g_n = 1 \\ \textit{modulo } \mathrm{B}_n \end{matrix} \right\}$$

- Braid group action (loops in the basespace) $\rightsquigarrow$ equivalence relation generated by:

$$(g_1, \ldots, g_i, g_{i+1}, \ldots, g_n) \sim (g_1, \ldots, g_i g_{i+1} g_i^{-1}, g_i, \ldots, g_n)$$

Connected components of the Hurwitz space

- Recall that we look for $\mathbb{Q}$-rational points of Hur.
  $\rightsquigarrow$ We need to find connected components defined over $\mathbb{Q}$.
  $\rightsquigarrow$ a problem on the way to (R)IGP! $\rightsquigarrow$ describe the connected components?
- Combinatorial description of connected components of $\mathrm{Hur}(G, n)$:

Proposition (Combinatorial description of the connected components of $\mathrm{Hur}(G, n)$)

$$\left\{ \begin{array}{c} \textit{Connected components} \\ \textit{of } \mathrm{Hur}(G, n) \end{array} \right\} \underset{\textit{non-canonical}}{\simeq} \left\{ \begin{array}{c} n - \textit{tuples } (g_1, \ldots, g_n) \in G^n \\ \textit{such that } g_1 \cdot g_2 \cdots g_n = 1 \\ \textit{modulo } \mathrm{B}_n \end{array} \right\}$$

- Braid group action (loops in the basespace) $\rightsquigarrow$ equivalence relation generated by:

$$(g_1, \ldots, g_i, g_{i+1}, \ldots, g_n) \sim (g_1, \ldots, g_i g_{i+1} g_i^{-1}, g_i, \ldots, g_n)$$

- The unordered list of conjugacy classes is an invariant: the *inertia canonical invariant*.

The ring of components

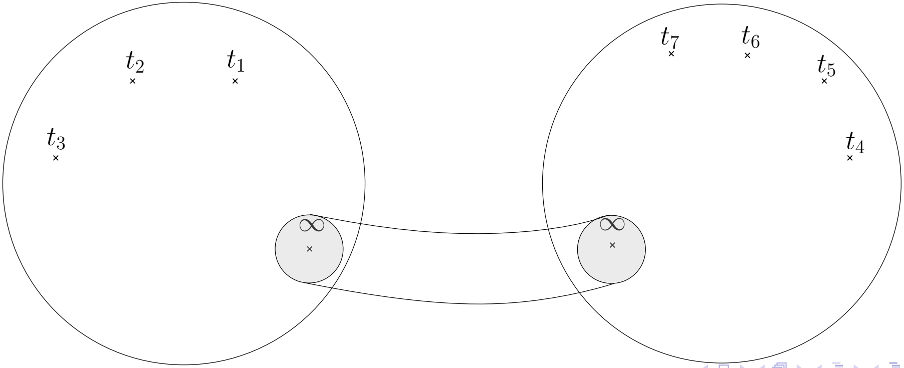
- Ellenberg, Venkatesh, Westerland 2016 (EVW):
  Covers are tuples + tuples may be concatenated $\rightsquigarrow$ Concatenate covers?

The ring of components

- Ellenberg, Venkatesh, Westerland 2016 (EVW):
  Covers are tuples + tuples may be concatenated $\rightsquigarrow$ Concatenate covers?
  Geometrically, this is a *gluing* operation:

Glue two projective lines together

$\Rightarrow$ get a single projective line with more branch points!



The ring of components

- Ellenberg, Venkatesh, Westerland 2016 (EVW):
  Covers are tuples + tuples may be concatenated $\rightsquigarrow$ Concatenate covers‽
- $\rightsquigarrow$ a graded monoid Comp_G of components of $\bigsqcup_{n \in \mathbb{N}} \mathrm{Hur}(G, n)$ w/ product:

$$(g_1, \ldots, g_n) \cdot (g'_1, \ldots, g'_{n'}) = (g_1, \ldots, g_n, g'_1, \ldots, g'_{n'});$$

  Important idea: the number of branch points varies $\rightsquigarrow$ asymptotical behaviour?

The ring of components

- Ellenberg, Venkatesh, Westerland 2016 (EVW):
  Covers are tuples + tuples may be concatenated $\rightsquigarrow$ Concatenate covers?
- $\rightsquigarrow$ a graded monoid Comp_G of components of $\bigsqcup_{n\in\mathbb{N}} \mathrm{Hur}(G, n)$ w/ product:
  $$(g_1, \ldots, g_n) \cdot (g'_1, \ldots, g'_{n'}) = (g_1, \ldots, g_n, g'_1, \ldots, g'_{n'});$$
  Important idea: the number of branch points varies $\rightsquigarrow$ asymptotical behaviour?
- The **ring of components** is a graded k-algebra:
  $$R_G = k[\mathrm{Comp}_G].$$
  Elements = formal sums of connected components of Hurwitz spaces.

The ring of components

- Ellenberg, Venkatesh, Westerland 2016 (EVW):
  Covers are tuples + tuples may be concatenated $\rightsquigarrow$ Concatenate covers‽
- $\rightsquigarrow$ a graded monoid Comp_G of components of $\bigsqcup_{n\in\mathbb{N}} \mathrm{Hur}(G,n)$ w/ product:
  $$(g_1,\dots,g_n)\cdot(g'_1,\dots,g'_{n'}) = (g_1,\dots,g_n,g'_1,\dots,g'_{n'});$$
  Important idea: the number of branch points varies $\rightsquigarrow$ asymptotical behaviour?
- The **ring of components** is a graded k-algebra:
  $$R_G = k[\mathrm{Comp}_G].$$
  Elements = formal sums of connected components of Hurwitz spaces.
- **Why did they introduce this ring? What does it tell us?**

How is the ring of components useful?

The approach of Ellenberg, Venkatesh and Westerland

Algebraic properties of the ring of components	factorization lemma + central element $U \in R_G$ of finite-dimensional cokernel and kernel

How is the ring of components useful?

The approach of Ellenberg, Venkatesh and Westerland

Algebraic properties of the ring of components	factorization lemma + central element $U \in R_G$ of finite-dimensional cokernel and kernel
$\Downarrow$	
Geometric property of the Hurwitz space	homological stability: $\exists D$ such that for $n \gg 0$: $H_i(\mathrm{Hur}(G, n), k) \simeq H_i(\mathrm{Hur}(G, n + D), k)$

How is the ring of components useful?

The approach of Ellenberg, Venkatesh and Westerland

Algebraic properties of the ring of components	factorization lemma + central element $U \in R_G$ of finite-dimensional cokernel and kernel
$\Downarrow$	
Geometric property of the Hurwitz space	homological stability: $\exists D$ such that for $n \gg 0$: $H_i(\mathrm{Hur}(G,n),k) \simeq H_i(\mathrm{Hur}(G,n+D),k)$
$\Downarrow$	
Field-theoretic result	asymptotic count of quadratic extensions of $\mathbb{F}_{p^n}(T)$ satisfying a condition on their class group, as $n \to \infty$ (related to the Cohen-Lenstra conjecture)

How is the ring of components useful?

The approach of Ellenberg, Venkatesh and Westerland

Algebraic properties of the ring of components	factorization lemma + central element $U \in R_G$ of finite-dimensional cokernel and kernel
$\Downarrow$	
Geometric property of the Hurwitz space	homological stability: $\exists D$ such that for $n \gg 0$: $H_i(\mathrm{Hur}(G, n), k) \simeq H_i(\mathrm{Hur}(G, n + D), k)$
$\Downarrow$	
Field-theoretic result	asymptotic count of quadratic extensions of $\mathbb{F}_{p^n}(T)$ satisfying a condition on their class group, as $n \to \infty$ (related to the Cohen-Lenstra conjecture)

Generalize this approach for different questions?

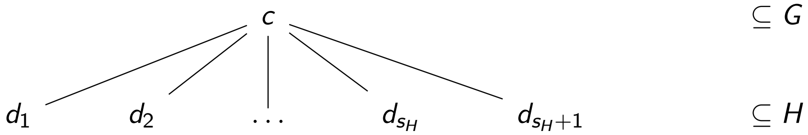
($\rightsquigarrow$ partial answers to IGP?)

An essential hypothesis of [EVW]: the non-splitting property (NSP)

- c = a conjugacy class of G;
- R_G^c = ring of components but only tuples $(g_1, \ldots, g_n) \in c^n$ s.t. $g_1 \cdots g_n = 1$;

An essential hypothesis of [EVW]: the non-splitting property (NSP)

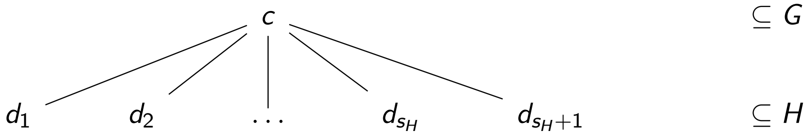
- c = a conjugacy class of G;
- R^c_G = ring of components but only tuples $(g_1, \ldots, g_n) \in c^n$ s.t. $g_1 \cdots g_n = 1$;
- If $H \subseteq G$: the set $c \cap H$ may consist of several conjugacy classes of H:



Call s_H the *splitting number*.

An essential hypothesis of [EVW]: the non-splitting property (NSP)

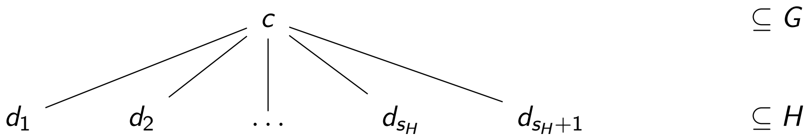
- c = a conjugacy class of G;
- R_G^c = ring of components but only tuples $(g_1, \dots, g_n) \in c^n$ s.t. $g_1 \cdots g_n = 1$;
- If $H \subseteq G$: the set $c \cap H$ may consist of several conjugacy classes of H:



  Call s_H the *splitting number*.
- If $s_H = 0$ for all subgroups H that intersect c, the couple (G, c) satisfies the **non-splitting property** (NSP): $c \cap H$ is a conjugacy class.

An essential hypothesis of [EVW]: the non-splitting property (NSP)

- c = a conjugacy class of G;
- R_G^c = ring of components but only tuples $(g_1, \ldots, g_n) \in c^n$ s.t. $g_1 \cdots g_n = 1$;
- If $H \subseteq G$: the set $c \cap H$ may consist of several conjugacy classes of H:



Call s_H the *splitting number*.

- If $s_H = 0$ for all subgroups H that intersect c, the couple (G, c) satisfies the **non-splitting property** (NSP): $c \cap H$ is a conjugacy class.
- Under the NSP, [EVW] establish homological stability.
  For H_0: the nb. of components $HF(n)$ stops increasing with n :

$$HF(n) = O(1)$$

The ring of components is geometrical

- One difference: [EVW] study covers of $\mathbb{A}^1(\mathbb{C})$, we study covers of $\mathbb{P}^1(\mathbb{C})$.
  $\rightsquigarrow$ R_G^c is a finitely generated **commutative** k-algebra...

The ring of components is geometrical

- One difference: [EVW] study covers of $\mathbb{A}^1(\mathbb{C})$, we study covers of $\mathbb{P}^1(\mathbb{C})$.
  $\rightsquigarrow$ R_G^c is a finitely generated **commutative** k-algebra...
- We can define the space $\mathrm{Proj}\, R_G^c$:
  $\rightsquigarrow$ What does it look like?
  $\rightsquigarrow$ What information does it carry about covers?

The ring of components is geometrical

- One difference: [EVW] study covers of $\mathbb{A}^1(\mathbb{C})$, we study covers of $\mathbb{P}^1(\mathbb{C})$.
  $\rightsquigarrow$ R_G^c is a finitely generated **commutative** k-algebra...
- We can define the space $\operatorname{Proj} R_G^c$:
  $\rightsquigarrow$ What does it look like?
  $\rightsquigarrow$ What information does it carry about covers?
- We can reformulate EVW's result in geometrical terms:

$$\begin{aligned}\text{Non-splitting Property} \quad &\Rightarrow \quad \text{Homological stability for } H_0 \\ &\Rightarrow \quad \operatorname{Proj} R_G^c \text{ has dimension } 0.\end{aligned}$$

- I wanted to generalize this geometrical version of their theorem.

Generalizations

Theorem (S. 22)

The dimension of $\mathrm{Proj}\, R_G^c$ *is equal to* $\max_{H \subseteq G} s_H$.

Generalizations

Theorem (S. 22)

The dimension of $\mathrm{Proj}\, R_G^c$ *is equal to* $\max_{H \subseteq G} s_H$.

More precise statement that deals with each subgroup H:
I attribute to H a subset $\gamma(H)$ of $\mathrm{Proj}\, R_G^c(k)$ ($\rightsquigarrow$partition). Then:

Group theory	Splitting number s_H
	$\Vert$
Geometry	Dimension of $\gamma(H)$
	$\Vert$
Combinatorics	Smallest d such that there are $O(n^d)$ components of group H

Generalizations

Theorem (S. 22)

The dimension of $\mathrm{Proj}\, R_G^c$ *is equal to* $\max_{H \subseteq G} s_H$.

More precise statement that deals with each subgroup H:
I attribute to H a subset $\gamma(H)$ of $\mathrm{Proj}\, R_G^c(k)$ ($\rightsquigarrow$partition). Then:

Group theory	Splitting number s_H
	$\Vert$
Geometry	Dimension of $\gamma(H)$
	$\Vert$
Combinatorics	Smallest d such that there are $O(n^d)$ components of group H

$\rightsquigarrow$ quantitative version for H_0 of "NSP $\Leftrightarrow$ homological stability"

Generalizations

Theorem (S. 22)

The dimension of $\mathrm{Proj}\, R_G^c$ *is equal to* $\max_{H \subseteq G} s_H$.

More precise statement that deals with each subgroup H:
I attribute to H a subset $\gamma(H)$ of $\mathrm{Proj}\, R_G^c(k)$ ($\rightsquigarrow$partition). Then:

Group theory	Splitting number s_H
	$\Vert$
Geometry	Dimension of $\gamma(H)$
	$\Vert$
Combinatorics	Smallest d such that there are $O(n^d)$ components of group H

$\rightsquigarrow$ quantitative version for H_0 of "NSP $\Leftrightarrow$ homological stability"
$\rightsquigarrow$ what about higher homology?

The example of the symmetric group

$G = \mathfrak{S}_d$, c is the conjugacy class of transpositions.
Studied by Hurwitz/Clebsch/Lüroth around 1872.

- We can describe R_G^c entirely:

$$R_G^c = \frac{k[(X_{ij})_{1 \leq i < j \leq d}]}{(X_{ij}X_{jk} = X_{ik}X_{jk} = X_{ij}X_{ik})_{1 \leq i < j < k \leq d}}$$

The example of the symmetric group

$G = \mathfrak{S}_d$, c is the conjugacy class of transpositions.
Studied by Hurwitz/Clebsch/Lüroth around 1872.

- We can describe R_G^c entirely:

$$R_G^c = \frac{k[(X_{ij})_{1 \le i < j \le d}]}{(X_{ij}X_{jk} = X_{ik}X_{jk} = X_{ij}X_{ik})_{1 \le i < j < k \le d}}$$

- Full description of $\mathrm{Proj}\, R_G^c(k)$, as a subvariety of $\mathbb{P}^{\frac{d(d-1)}{2}-1}(k)$ of dimension $\lfloor d/2 \rfloor - 1$:
  - a vertex e_A for each $A \subseteq \{1, \ldots, d\}$ of size at least 2 $\rightsquigarrow \mathfrak{S}_A$

The example of the symmetric group

$G = \mathfrak{S}_d$, c is the conjugacy class of transpositions.
Studied by Hurwitz/Clebsch/Lüroth around 1872.

- We can describe R_G^c entirely:

$$R_G^c = \frac{k[(X_{ij})_{1 \leq i < j \leq d}]}{(X_{ij}X_{jk} = X_{ik}X_{jk} = X_{ij}X_{ik})_{1 \leq i < j < k \leq d}}$$

- Full description of $\operatorname{Proj} R_G^c(k)$, as a subvariety of $\mathbb{P}^{\frac{d(d-1)}{2}-1}(k)$ of dimension $\lfloor d/2 \rfloor - 1$:
  - a vertex e_A for each $A \subseteq \{1, \ldots, d\}$ of size at least 2 $\rightsquigarrow \mathfrak{S}_A$
  - a line joining e_A, e_B when A, B are disjoint $\rightsquigarrow \mathfrak{S}_A \times \mathfrak{S}_B$

The example of the symmetric group

$G = \mathfrak{S}_d$, c is the conjugacy class of transpositions.
Studied by Hurwitz/Clebsch/Lüroth around 1872.

- We can describe R_G^c entirely:

$$R_G^c = \frac{k[(X_{ij})_{1 \leq i < j \leq d}]}{(X_{ij}X_{jk} = X_{ik}X_{jk} = X_{ij}X_{ik})_{1 \leq i < j < k \leq d}}$$

- Full description of $\operatorname{Proj} R_G^c(k)$, as a subvariety of $\mathbb{P}^{\frac{d(d-1)}{2}-1}(k)$ of dimension $\lfloor d/2 \rfloor - 1$:
  - a vertex e_A for each $A \subseteq \{1, \ldots, d\}$ of size at least 2 $\rightsquigarrow \mathfrak{S}_A$
  - a line joining e_A, e_B when A, B are disjoint $\rightsquigarrow \mathfrak{S}_A \times \mathfrak{S}_B$
  - a plane containing e_A, e_B, e_C when A, B, C are disjoint $\rightsquigarrow \mathfrak{S}_A \times \mathfrak{S}_B \times \mathfrak{S}_C$
  - etc.

Example of $\mathfrak{S}_3$

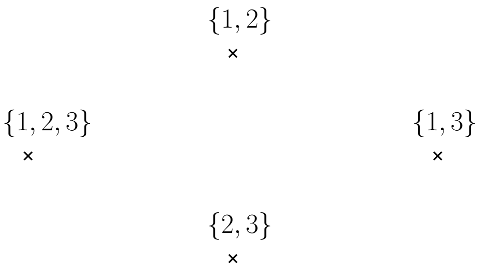


Figure: $\mathrm{Proj}\, R_G^c(k)$ in the case $G = \mathfrak{S}_3$, $c = \{\text{transpositions}\}$

Only points, no lines
$\rightsquigarrow$ dimension 0
$\rightsquigarrow$ NSP/homological stability/etc.!

Example of $\mathfrak{S}_4$

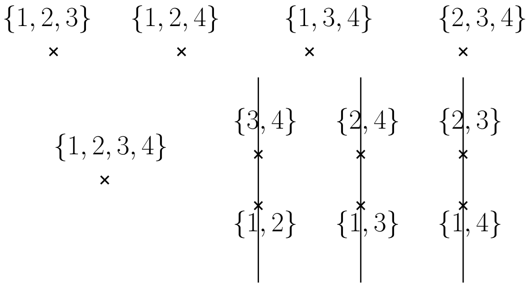
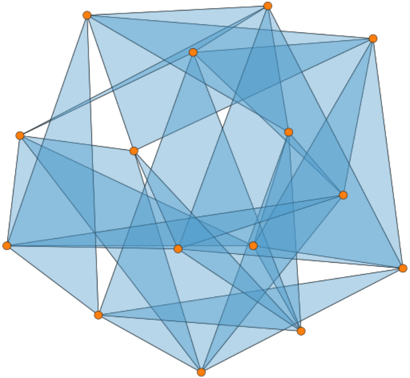


Figure: $\operatorname{Proj} R_G^c(k)$ in the case $G = \mathfrak{S}_4$, $c = \{\text{transpositions}\}$

dimension 1 $\rightsquigarrow$ it gets more interesting!
(the drawing is very schematic since I may not draw in 5D)

Example of $\mathfrak{S}_6$?

To observe dimension 2, we need to go to $d = 6$. Problem: there is much to draw $(1 + 6 + 15 + 40 + 15 = 77$ vertices, $15 + 20 + 80 + 45 = 160$ lines, 15 planes).
Here is what I obtained when I tried to draw *just the part with subsets of size* 2 (i.e. the 15 planes, represented as triangles) using a tool for drawing simplicial sets by simulating springs:



A general result

Terminology: By c-**subgroup** of G, I mean "subgroup generated by elements of c".
Non-splitters are c-subgroups H such that $s_H = 0$, i.e. $c \cap H$ is a conjugacy class of H.

A general result

Terminology: By c-**subgroup** of G, I mean "subgroup generated by elements of c".
Non-splitters are c-subgroups H such that $s_H = 0$, i.e. $c \cap H$ is a conjugacy class of H.

Theorem (S. 22)

Assume:

- *All c-subgroups of G are direct products of non-splitters:* $H = H_1 \times \ldots \times H_r$, *Moreover* $c \cap H = \bigsqcup_{i=1}^{r} c \cap H_i$
- *If H is a non-splitter, there is at most one component of group H with n branch points*

A general result

Terminology: By c-**subgroup** of G, I mean "subgroup generated by elements of c".
Non-splitters are c-subgroups H such that $s_H = 0$, i.e. $c \cap H$ is a conjugacy class of H.

Theorem (S. 22)

Assume:

- *All c-subgroups of G are direct products of non-splitters:* $H = H_1 \times \ldots \times H_r$,
  Moreover $c \cap H = \bigsqcup_{i=1}^{r} c \cap H_i$
- *If H is a non-splitter, there is at most one component of group H with n branch points*

Then we have a full description of $\mathrm{Proj}\, R_G^c(k)$*:*

- *For each non-splitter* $H \rightsquigarrow$ *a vertex* $e(H)$*;*

A general result

Terminology: By c-**subgroup** of G, I mean "subgroup generated by elements of c".
Non-splitters are c-subgroups H such that $s_H = 0$, i.e. $c \cap H$ is a conjugacy class of H.

Theorem (S. 22)

Assume:

- *All c-subgroups of G are direct products of non-splitters:* $H = H_1 \times \ldots \times H_r$, *Moreover* $c \cap H = \bigsqcup_{i=1}^{r} c \cap H_i$
- *If H is a non-splitter, there is at most one component of group H with n branch points*

Then we have a full description of $\operatorname{Proj} R_G^c(k)$*:*

- *For each non-splitter* $H \rightsquigarrow$ *a vertex* $e(H)$*;*
- *For each product* $H_1 \times \ldots \times H_r \rightsquigarrow$ *the* $(n-1)$*-dimensional span of* $e(H_i), i = 1, \ldots, r$.

A general result

Terminology: By c-**subgroup** of G, I mean "subgroup generated by elements of c".
Non-splitters are c-subgroups H such that $s_H = 0$, i.e. $c \cap H$ is a conjugacy class of H.

Theorem (S. 22)

Assume:

- *All c-subgroups of G are direct products of non-splitters: $H = H_1 \times \ldots \times H_r$, Moreover $c \cap H = \bigsqcup_{i=1}^{r} c \cap H_i$*
- *If H is a non-splitter, there is at most one component of group H with n branch points*

Then we have a full description of $\mathrm{Proj}\, R_G^c(k)$:

- *For each non-splitter $H \rightsquigarrow$ a vertex $e(H)$;*
- *For each product $H_1 \times \ldots \times H_r \rightsquigarrow$ the $(n-1)$-dimensional span of $e(H_i), i = 1, \ldots, r$.*

In particular, dimension = maximal number of direct factors in a c-subgroup, minus one.

Rationality and products

- Arithmetic aspects: Galois action on $\mathrm{Hur}(G, n) \rightsquigarrow$ components defined over $\mathbb{Q}$?
- **Arithmetic version** of the ring of components?

Rationality and products

- Arithmetic aspects: Galois action on $\mathrm{Hur}(G, n) \rightsquigarrow$ components defined over $\mathbb{Q}$?
- **Arithmetic version** of the ring of components?
- How does this notion of rationality interact with products?
  If x, y are components defined over $\mathbb{Q}$, is the product xy defined over $\mathbb{Q}$?

Rationality and products

- Arithmetic aspects: Galois action on $\mathrm{Hur}(G, n) \rightsquigarrow$ components defined over $\mathbb{Q}$?
- **Arithmetic version** of the ring of components?
- How does this notion of rationality interact with products?
  If x, y are components defined over $\mathbb{Q}$, is the product xy defined over $\mathbb{Q}$?
  $\rightsquigarrow$ Unclear to me.

Rationality and products

- Arithmetic aspects: Galois action on $\mathrm{Hur}(G, n) \rightsquigarrow$ components defined over $\mathbb{Q}$?
- **Arithmetic version** of the ring of components?
- How does this notion of rationality interact with products?
  If x, y are components defined over $\mathbb{Q}$, is the product xy defined over $\mathbb{Q}$?
  $\rightsquigarrow$ Unclear to me.
- However:

Theorem (Cau 12)

For components x, y defined over $\mathbb{Q}$,

$$\left\{ x^g y^{g'} \,\middle|\, g, g' \in G \right\} \text{ is stable under the action of } \mathrm{Gal}\left(\overline{\mathbb{Q}} \mid \mathbb{Q}\right).$$

A subring of rational components

Let:

$$R'_G = \{x \in R_G \mid x \text{ invariant under } \mathrm{Inn}(G)\}.$$

$\leadsto$ Subring of R_G generated by sums $\sum_{g \in G} x^g$ (where x = component).

A subring of rational components

Let:

$$R'_G = \{x \in R_G \mid x \text{ invariant under } \mathrm{Inn}(G)\}.$$

$\rightsquigarrow$ Subring of R_G generated by sums $\sum_{g \in G} x^g$ (where x = component).

Proposition (S. 22)

The Galois action $\mathrm{Gal}(\overline{\mathbb{Q}} \mid \mathbb{Q})$ on R'_G is compatible with the product:

$$\sigma.(xy) = (\sigma.x)(\sigma.y) \textit{ for } \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}} \mid \mathbb{Q}).$$

A subring of rational components

Let:

$$R'_G = \{x \in R_G \mid x \text{ invariant under } \mathrm{Inn}(G)\}.$$

$\rightsquigarrow$ Subring of R_G generated by sums $\sum_{g \in G} x^g$ (where x = component).

Proposition (S. 22)

The Galois action $\mathrm{Gal}(\overline{\mathbb{Q}} \mid \mathbb{Q})$ *on* R'_G *is compatible with the product:*

$$\sigma.(xy) = (\sigma.x)(\sigma.y) \text{ for } \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}} \mid \mathbb{Q}).$$

$\rightsquigarrow$ In R'_G, products of elements defined over $\mathbb{Q}$ are defined over $\mathbb{Q}$.

$\rightsquigarrow$ We can define a subring $R'_{G,\mathbb{Q}}$ of elements of R'_G defined over $\mathbb{Q}$.

A subring of rational components

Let:

$$R'_G = \{x \in R_G \mid x \text{ invariant under } \mathrm{Inn}(G)\}.$$

$\rightsquigarrow$ Subring of R_G generated by sums $\sum_{g \in G} x^g$ (where x = component).

Proposition (S. 22)

The Galois action $\mathrm{Gal}(\overline{\mathbb{Q}} \mid \mathbb{Q})$ *on* R'_G *is compatible with the product:*

$$\sigma.(xy) = (\sigma.x)(\sigma.y) \text{ for } \sigma \in \mathrm{Gal}(\overline{\mathbb{Q}} \mid \mathbb{Q}).$$

$\rightsquigarrow$ In R'_G, products of elements defined over $\mathbb{Q}$ are defined over $\mathbb{Q}$.

$\rightsquigarrow$ We can define a subring $R'_{G,\mathbb{Q}}$ of elements of R'_G defined over $\mathbb{Q}$.

$\rightsquigarrow$ Arithmetical version of R_G! **What can we do with it?**

Current investigations

- Extend our results for R_G to $R'_{G,\mathbb{Q}}$? $\rightsquigarrow$ obtain arithmetical versions of our results;

Current investigations

- Extend our results for R_G to $R'_{G,\mathbb{Q}}$? $\rightsquigarrow$ obtain arithmetical versions of our results;
- Study examples of components (not) defined over $\mathbb{Q}$ using our methods;

Current investigations

- Extend our results for R_G to $R'_{G,\mathbb{Q}}$? $\rightsquigarrow$ obtain arithmetical versions of our results;
- Study examples of components (not) defined over $\mathbb{Q}$ using our methods;
- Study $\operatorname{Proj} R_G$ further $\rightsquigarrow$ relate its geometry to field extensions;

Current investigations

- Extend our results for R_G to $R'_{G,\mathbb{Q}}$? $\rightsquigarrow$ obtain arithmetical versions of our results;
- Study examples of components (not) defined over $\mathbb{Q}$ using our methods;
- Study $\operatorname{Proj} R_G$ further $\rightsquigarrow$ relate its geometry to field extensions;
- Understand higher homology: the methods of [EVW] do not generalize.

Current investigations

- Extend our results for R_G to $R'_{G,\mathbb{Q}}$? $\rightsquigarrow$ obtain arithmetical versions of our results;
- Study examples of components (not) defined over $\mathbb{Q}$ using our methods;
- Study $\mathrm{Proj}\, R_G$ further $\rightsquigarrow$ relate its geometry to field extensions;
- Understand higher homology: the methods of [EVW] do not generalize.

Main bibliography:

- Jordan S. Ellenberg, Akshay Venkatesh and Craig Westerland. "Homological stability for Hurwitz spaces and the Cohen-Lenstra conjecture over function fields". In: *Annals of Mathematics 183.3 (2016)*, pp. 729–786.
- Orlando Cau. "Delta-composantes des espaces de modules de revêtements". In: *Journal de Théorie des Nombres de Bordeaux 24.3 (2012)*, pp. 557–582.