Mathematisches Forschungsinstitut Oberwolfach

Report No. 42/2023

DOI: 10.4171/OWR/2023/42

MFO-RIMS Tandem Workshop – Arithmetic Homotopy and Galois Theory

Organized by
B. Collas, Kyoto
P. Dèbes, Villeneuve d'Ascq
Y. Hoshi, Kyoto
A. Mézard, Paris

24 September – 29 September, 2023

Abstracts

Covers of $\mathbb{P}^1$ and their moduli: where arithmetic, geometry and combinatorics meet

Béranger Seguin

During the last fifty years, the theory of finite branched covers of the projective line has played a major role in inverse Galois theory. The main reason behind this success is that this theory makes it possible to use topological and geometric arguments to study Galois theory over function fields (with consequences over number fields because of Hilbert's irreducibility theorem). Moreover, the topological objects involved admit combinatorial descriptions — this has allowed computational approaches to shed new light on various aspects of inverse Galois theory.

In this report, we present two contributions: the first one is the description of combinatorial objects generalizing *dessins d'enfants* to covers of the line with arbitrary numbers of branch points, the second one is a patching result over number fields for components of Hurwitz spaces, i.e. irreducible families of covers.

For the whole report, we fix a finite group G and an integer n.

1. Covers of the line

We fix a set $\underline{t} = \{t_1, \ldots, t_n\}$ of n distinct points of the complex projective line $\mathbb{P}^1(\mathbb{C})$ (which we call a *configuration*) and a basepoint $t_0 \in \mathbb{P}^1(\mathbb{C}) \setminus \underline{t}$. We start by recalling some terminology to avoid any ambiguity.

1.1. ***G*-covers.** In this report, a *cover* (branched at $\underline{t}$) always refers to a finite covering map $p : Y \to \mathbb{P}^1(\mathbb{C}) \setminus \underline{t}$. A *marked* cover comes with a point in the fiber $p^{-1}(t_0)$. A *G-cover* comes with a group morphism $G \to \mathrm{Aut}(p)$ inducing a simply transitive action of G on $p^{-1}(t_0)$. We do **not** require connectedness. *Connected* G-covers are Galois covers with automorphism group isomorphic to G.

The *monodromy morphism* of a marked G-cover is a group morphism $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \underline{t}, t_0) \to G$, which is surjective if and only if the cover is connected. Its image is the *monodromy group* of the cover. Each group morphism $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \underline{t}, t_0) \to G$ is the monodromy morphism of a marked G-cover, unique up to isomorphism.

Since the fundamental group of $\mathbb{P}^1 \setminus \underline{t}$ is generated by loops $\gamma_1, \ldots, \gamma_n$ subject to the sole relation $\gamma_1 \cdots \gamma_n = 1$, isomorphism classes of marked G-covers branched at $\underline{t}$ correspond to n-tuples $(g_1, \ldots, g_n)$ of elements of G (the *monodromy elements*) satisfying $g_1 \cdots g_n = 1$. Connectedness corresponds to the condition that the monodromy elements generate G.

1.2. **Generalized dessins.** In the case $n = 3$, a combinatorial model of covers of $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ has been introduced in [4] under the name *dessins d'enfants*. The case $n = 3$ is special in two ways:

(1) Since $\mathrm{PSL}_2(\mathbb{C})$ acts 3-transitively on points, all choices of $\underline{t}$ are equivalent.
(2) This case is universal for the study of algebraic curves: by Belyĭ's theorem, every curve defined over $\bar{\mathbb{Q}}$ covers $\mathbb{P}^1(\mathbb{C})$ with at most 3 branch points.

However, if one is interested not in algebraic curves, but in covers (*morphisms* between curves) with arbitrary numbers of branch points, which many applications in inverse Galois theory involve, then this description is not enough. For example, there are no connected G-covers when $n = 3$ and G is not 2-generated.

In ongoing work[1], we define and study a notion of "generalized dessins". These objects may be described as $(n-1)$-partite "rainbow-colored" hypermaps (instead of being edges, the "hyperedges" are $(n-1)$-gones with one vertex of each color, and there are $n-1$ colors) embedded on surfaces. One hope is that, starting with this description, a program to describe the Galois action on covers combinatorially is developed, in the spirit of Grothendieck-Teichmüller theory which has basically (although this is a vast simplification) come out of the case $n = 3$.

Here is an example:

Write down the cycles corresponding to the appearance order of the hyperedges during a counterclockwise rotation around each white vertex. The product of these cycles defines a permutation $\sigma_\circ = (152)(364)$ of the hyperedges. Doing the same for crossed and black vertices yields $\sigma_\otimes = (14)$ and $\sigma_\bullet = (16)(24)(35)$.

[1] At the moment, this work has only been made public as a series of blog posts, accessible at the following URL: `https://lebarde.alwaysdata.net/blog/2023/dessins-1/`.

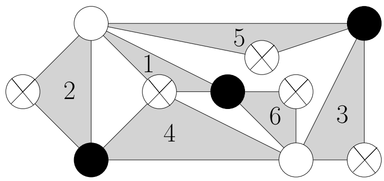

FIGURE 1. A generalized dessin corresponding to the case $n = 4$, where we have labeled the hyperedges (grey triangles)

Finally, let $\sigma_\infty = (\sigma_\circ \sigma_\otimes \sigma_\bullet)^{-1} = (13)(45)$, whose four cycles correspond to the four connected components of the complement of the dessin (i.e. the white areas) – depending on which component the •–○ boundary of a given hyperedge touches. The permutations $\sigma_\circ, \sigma_\otimes, \sigma_\bullet, \sigma_\infty$ are the monodromy elements of a cover: this dessin corresponds to a non-Galois connected cover of degree 6 of the projective line branched at four points. Its monodromy group is the subgroup of $\mathfrak{S}_6$ generated by $\sigma_\circ$, $\sigma_\otimes$, and $\sigma_\bullet$, which is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathfrak{S}_4$. This cover has genus 0 (it is embedded in this page!).

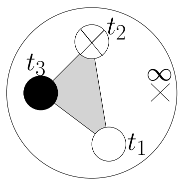

FIGURE 2. A triangle, drawn on $\mathbb{P}^1(\mathbb{C}) \setminus \{t_1, t_2, t_3, \infty\}$, whose preimage under a covering map gives the corresponding dessin.

2. Patching components of Hurwitz spaces over number fields

In this section, we present Hurwitz spaces, their components, and the gluing of components, and we give number-theoretical applications of these objects. The original results in this section are all in [6, 8, 7]. We fix a number field K.

2.1. **Hurwitz spaces.** Riemann's existence theorem implies that covers form a category equivalent to that of *algebraic covers*, i.e. generically étale finite morphisms from a smooth curve to $\mathbb{P}^1_{\mathbb{C}}$. Since smooth curves are determined by their function fields, connected G-covers correspond to Galois extensions of $\mathbb{C}(T)$ with Galois group G. If a G-cover is moreover defined over K, it corresponds to a *regular* extension $F|K(T)$, where regular means that $F \cap \bar{K} = K$.

There is a $\mathbb{Z}[1/|G|]$-scheme $\mathrm{Hur}^*_{G,n}$, the *Hurwitz space*, whose $\mathbb{C}$-points correspond to marked G-covers branched at n distinct points. Moreover, K-points of this scheme correspond to regular Galois G-extension of $K(T)$ having an unramified prime of degree 1. To put it shortly, this turns instances of the inverse Galois problem into Diophantine problems: do Hurwitz spaces have rational points?

2.2. **Fields of definitions of concatenated components.** From now on, we call *component* a geometrically connected component of the Hurwitz space $\mathrm{Hur}^*_{G,n}$. Since a K-point must lie in a component defined over K, fields of definition of components are of special interest for inverse Galois theory: they tell us where to look for. There is a topological *gluing* operation on components, induced in terms of tuples by the concatenation:

$$(g_1, \ldots, g_n), (g'_1, \ldots, g'_{n'}) \mapsto (g_1, \ldots, g_n, g'_1, \ldots, g'_{n'}).$$

We denote by xy the component obtained by gluing two components x and y. The focus of [8] is the following question:

Problem 1. *Do components obtained by gluing components defined over K are also defined over K, and we give positive answers in various situations.*

Previous work on this question includes [1], where Cau obtains some positive results generalizing those of [3]. The following result is [8, Theorem 5.4]:

Theorem 2. *Let x, y be components defined over K with respective monodromy groups H_1, H_2 ($\subseteq G$). Let $H = \langle H_1, H_2 \rangle$. Then there is an element $\gamma \in H$ such that $H = \langle H_1, H_2^\gamma \rangle$ and such that xy^γ is defined over K.*

The proof of the theorem is in three steps:

Step 1 – construct infinitely many linearly disjoint extensions of K over which x and y have points. Take arbitrary geometric points in the components x and y lying above a K-rational configuration, and denote by K_1 the smallest Galois extension of K over which they are rational. By Hilbert's irreducibility theorem, there is a K-rational configuration above which the fibers of x and y are both irreducible over K_1. Choose arbitrary geometric points in the fibers of x and y above $\underline{t}$. Let K_2 be the smallest Galois extension of K over which these points are both rational. By irreducibility of the fibers, K_2 and K_1 are linearly disjoint over K. Iterate this process to define an infinite sequence $K_1, K_2, \ldots$ of pairwise linearly disjoint extensions of K such that for all $i \geq 1$, the components x, y both have K_i-points, denoted respectively f_i and g_i.

Step 2 – patching. See f_i and g_i as covers over the complete valued field $K_i((X))$. Use the algebraic variant of Harbater's theory of patching (cf. [5]) to patch them into a cover defined over $K_i((X))$ with monodromy group H. By a result of Cau [1, Prop. 3.9], the patched cover is in a component c_i of the form $x^{\gamma'_i} y^{\gamma_i}$.

Step 3 – pigeonhole. Since there are finitely many components of the form $x^{\gamma'} y^{\gamma}$, there are distinct i, i' such that $c_i = c_{i'}$. The component $c_i = c_{i'}$ is defined over $\bar{K} \cap K_i((X)) \cap K_{i'}((X)) = K$. Finally, conjugate c_i by $(\gamma'_i)^{-1}$ to ensure $\gamma' = 1$. □

This theorem may be used to construct components defined over $\mathbb{Q}$ with relatively few branch points compared to those constructed in [1]:

Corollary 3 ([7, Proposition 8.4.8])**.** *If G is generated by elements $g_1, \ldots, g_n$ among which $m(i)$ elements are of order i, there is a component defined over $\mathbb{Q}$ of the Hurwitz space of connected G-covers whose number of branch points is $2m(2) + \sum_{i \geq 3} m(i)\varphi(i)$, where φ denotes Euler's totient function.*

For example, if the group G is generated by two elements with orders in $\{2, 3, 4, 6\}$, then there are components defined over $\mathbb{Q}$ of connected G-covers with four branch points (of Harbater-Mumford type). This applies to the Mathieu group M_{23} and to the group $\mathrm{PSL}_2(16) \rtimes \mathbb{Z}/2\mathbb{Z}$.

2.3. **The use of gluing for enumerative problems, and extensions of $\mathbb{F}_q(T)$.** Besides allowing to construct components with small fields of definition, the gluing operation also helps in estimating the asymptotical homology of Hurwitz spaces, which is key to the study of Malle's conjecture over function fields over finite fields. We refer to [2] or to Westerland's talk in the present volume for additional details.

The gluing operation on components of Hurwitz spaces induces a ring structure on the set of formal sums of components (the "ring of components"). In [6], we studied this ring closely in order to compute both the exponent (the "splitting number") and the leading coefficient of the asymptotical number of components as the number of branch points increases. This estimate has been applied in the updated version of [2] to the question of the distribution of G-extensions of $\mathbb{F}_q(T)$.

References

[1] O. Cau, *Delta-composantes des espaces de modules de revêtements*, Journal de Théorie des Nombres de Bordeaux **24.3** (2012), 557–582.

[2] J. S. Ellenberg, T. Tran, and C. Westerland, *Fox-Neuwirth- Fuks cells, quantum shuffle algebras, and Malle's conjecture for function fields (v2)*, arXiv: 1701.04541v2 (2023).

[3] P. Dèbes and M. Emsalem. *Harbater-Mumford Components and Towers of Moduli Spaces*, Journal of the Institute of Mathematics of Jussieu **5** (2006), 351–371.

[4] A. Grothendieck, *Esquisse d'un programme* (1984). Published in *Geometric Galois Actions Vol. 1*, editors: P. Lochak, L. Schneps (1997), pp. 5-48 (English translation: pp. 243-283).

[5] D. Haran and H. Völklein, *Galois groups over complete valued fields*, Israel Journal of Mathematics **93** (1996), 9–27.

[6] B. Seguin. *The Geometry of Rings of Components of Hurwitz Spaces*, arXiv: 2210.12793 (2022).

[7] B. Seguin, *Géométrie et arithmétique des composantes des espaces de Hurwitz*, PhD thesis, Université de Lille (2023).

[8] B. Seguin, *Fields of Definition of Components of Hurwitz Spaces*, arXiv: 2303.05903 (2023).