Fields of Definition of Components of Hurwitz Spaces

Béranger Seguin*

Abstract: For a fixed finite group G, we study the fields of definition of geometrically irreducible components of Hurwitz spaces parametrizing marked branched G-covers of the projective line. The primary focus is on determining whether components obtained by "gluing" two components, both defined over a number field K, remain defined over K. The article presents a list of situations in which a positive answer is obtained. As an application, components defined over $\mathbb{Q}$ of "small" dimension are constructed for all groups G, using patching methods.

MSC 2020: 14H10 · 14H30 · 14D10 · 12F12

Contents

1. Introduction

Context. Let G be a finite group, and let K be a field of characteristic zero. Hurwitz spaces are moduli spaces parametrizing branched G-covers of $\mathbb{P}^1$, and their K-points are tightly related to the inverse Galois problem for G over $K(T)$; see [Fri77, FV91, RW06].

When K is algebraically closed, the study of such covers reduces to topology by Riemann's existence theorem: the $\mathbb{C}$-points of Hurwitz spaces correspond to isomorphism classes of topological G-covers of punctured Riemann spheres. A classical topological construction allows one to glue two marked covers — one with r_1 branch points and another with r_2 branch points — into a single marked cover with $r_1 + r_2$ branch points. This gluing operation plays a central role in [EVW16].

When K is instead a complete non-Archimedean valued field, Harbater has introduced an analogous *patching* operation, which allows one to construct connected covers of $\mathbb{P}^1_K$ with a given automorphism group by patching together covers with smaller automorphism groups, leading to a positive answer to the inverse Galois problem over $K(T)$; see [Liu95, HV96, Har03].

For number theorists, the most interesting case is that of number fields, over which no gluing or patching operation is available. In this article, we focus not on G-covers themselves but on geometrically connected components of Hurwitz spaces (*families* of G-covers), and we study the possibility of gluing these components over a number field. Since finding components defined over $\mathbb{Q}$ is a crucial first step in finding rational points, this problem is connected to inverse Galois theory, and is a well-studied topic; see [FV91, DE06, Cau12].

*Universität Paderborn, Fakultät EIM, Institut für Mathematik, Warburger Str. 100, 33098 Paderborn, Germany. Email: bseguin@math.upb.de.

Main results. We fix a number field K, a finite group G, and we let $\mathrm{Comp}(G)$ be the set of geometrically connected components of the Hurwitz spaces classifying marked branched G-covers of $\mathbb{P}^1$, graded by the number of branch points of the covers in each component (this is defined more carefully in Subsection 2.2). The gluing operation turns the set $\mathrm{Comp}(G)$ into a graded monoid. A key question in understanding the arithmetic properties of the gluing operation is the following:

Question 1.1. Let $x, y \in \mathrm{Comp}(G)$ be components defined over K. Is the component $xy \in \mathrm{Comp}(G)$, obtained by gluing x and y, also defined over K?

Question 1.1 is the primary focus of this article. Our main results are positive answers in situations (i), (ii) and (iii) below:

Theorem 1.2. *Let $x, y \in \mathrm{Comp}(G)$ be components defined over K, and let H_1 and H_2 be the monodromy groups of the marked G-covers in x and y respectively. Let $H = \langle H_1, H_2 \rangle$. Then:*

(i) If $H_1 H_2 = H$, then the glued component xy is defined over K.

(ii) If every conjugacy class of H which is a local monodromy class of the covers in xy occurs at least at M branch points (for some constant M depending only on the group G), then xy is defined over K.

(iii) There exist $\gamma, \gamma' \in H$ satisfying $\left\langle H_1^{\gamma}, H_2^{\gamma'} \right\rangle = H$ and such that the component $x^{\gamma} y^{\gamma'}$, obtained by letting γ and γ' act on x and y and by gluing the resulting components, is defined over K.

In Section 2, we introduce the notation and the key objects. The three points of Theorem 1.2 are then proved in three corresponding sections:

- Theorem 1.2 (i) is the case $n = 2$ of the more general Theorem 3.3 (iii), whose proof combines techniques introduced in [Cau12] with properties of the Hurwitz action. In Section 3, we prove this result, and we propose applications in Subsection 3.3. Cases of interest include the situation where either H_1 or H_2 is normal in H, notably if $H_1 \supseteq H_2$ or if $H = H_1 \rtimes H_2$.

- Theorem 1.2 (ii) is the case $n = 2$ of the more general Theorem 4.7 (iii). The proof uses the *lifting invariant* from [EVW12, Woo21], which generalizes ideas of Conway, Parker, Fried and Völklein. In Section 4, we review this invariant and use it to prove our result.

- Theorem 1.2 (iii) is Theorem 5.4. Its proof is based on patching results over complete valued fields, following the algebraic approach of [HV96]. By patching covers over infinitely many complete valued fields, we obtain a result in the number field case. Section 5 deals with the proof of this theorem. An application is given in Example 5.5: we construct components defined over $\mathbb{Q}$ of connected G-covers with only four branch points when G is the Mathieu group M_{23}, without resorting to rigidity methods. This is generalized by Proposition 5.6.

Finally, in Section 6, we prove Propositions 6.1 and 6.2, which give another application of the ideas of Sections 3 and 4: we express the action of the absolute Galois group Γ_K on components with arbitrarily large numbers of branch points in terms of its action on components with few branch points. This is remindful of similar results at the level of covers (notably the faithfulness of the Galois action on covers with three branch points, which stems from Belyi's theorem, cf. [Sza09, Theorem 4.7.7]).

We do not know whether the answer to Question 1.1 is always positive. Finding counterexamples is hard as few tools are available to prove that a component is not defined over $\mathbb{Q}$. Moreover, Theorem 4.10 implies that the lifting invariant cannot detect counterexamples: the lifting invariant of a product of components defined over K is Galois-invariant, so products of components defined over K are indistinguishable from components defined over K from the point of view of this invariant.

Acknowledgments. This work was funded in part by the French ministry of research through a CDSN grant, and in part by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) — Project-ID 491392403 — TRR 358.

I thank my advisors Pierre Dèbes and Ariane Mézard for their support and their precious advice during my time as a PhD student, and the reporters of my thesis Jean-Marc Couveignes and Craig Westerland for providing valuable feedback, and the anonymous referee for their helpful comments.

2. Preliminaries

This section recalls the key definitions and results concerning (marked) G-covers, Hurwitz spaces and their components, and the Galois action on these objects. In Subsection 2.1, notational and terminological choices are presented. In Subsection 2.2, we introduce marked G-covers and Hurwitz spaces in both topological and algebraic settings. In Subsection 2.3, we relate the Galois action on marked G-covers and components of Hurwitz spaces to their fields of definitions.

2.1. Notation

In what follows, G is a finite group and K is a number field. Number fields are always equipped with an embedding into $\overline{\mathbb{Q}}$, which is itself identified with the subfield of $\mathbb{C}$ consisting of algebraic complex numbers. We denote by Γ_K the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}|K)$. The cyclotomic character is the group homomorphism $\chi\colon \Gamma_K \to \widehat{\mathbb{Z}}^\times$ determined by the Galois action on roots of unity: if $\zeta \in \overline{\mathbb{Q}}$ is an n-th root of unity and $\sigma \in \Gamma_K$, then $\sigma(\zeta) = \zeta^{\chi(\sigma) \bmod n}$.

2.1.1. Conventions. The cardinality of a set X is denoted by $|X|$. We write $g^h = hgh^{-1}$ for conjugation in a group. We denote by $\mathrm{ord}(g)$ the order of an element g in a finite group H. Similarly, if $c \subseteq H$ is a conjugacy class, we let $\mathrm{ord}(c)$ be the order of any element of c. If $g \in H$ and $\alpha \in \widehat{\mathbb{Z}}$ is a profinite integer, g^α is the well-defined element $g^{\alpha \bmod \mathrm{ord}(g)} \in H$.

Definition 2.1. A subset $c \subseteq G$ is *K-rational* if for every $g \in c$ and $\sigma \in \Gamma_K$ we have $g^{\chi(\sigma)} \in c$.

If $K = \mathbb{Q}$, we have $\mathrm{Im}(\chi) = \widehat{\mathbb{Z}}^\times$, so that a subset of G is $\mathbb{Q}$-rational if and only if it is closed under n-th powers for all n coprime with $|G|$. In contrast, if K contains all $|G|$-th roots of unity, then the image of χ is trivial modulo $|G|$ and every subset of G is K-rational. Examples of sets which are always K-rational include G, $G \setminus \{1\}$, as well as any subset of G consisting only of involutions.

2.1.2. Tuples. Tuples are denoted with underlined roman letters. Let $\underline{g} = (g_1, \ldots, g_n)$ be a tuple of elements of G. Then:

- Its *size* $\deg(\underline{g})$ is the number n of elements in the tuple.
- Its *group* $\left\langle \underline{g} \right\rangle$ is the subgroup of G generated by $g_1, \ldots, g_n$. If $\underline{g}_1, \ldots, \underline{g}_s$ are tuples, we denote by $\left\langle \underline{g}_1, \ldots, \underline{g}_s \right\rangle$ the subgroup of G generated by the subgroups $\left\langle \underline{g}_1 \right\rangle, \ldots, \left\langle \underline{g}_s \right\rangle$.
- The *product* of $\underline{g}$ is $\pi\underline{g} = g_1 g_2 \cdots g_n \in G$. We say that $\underline{g}$ is a *product-one tuple* if $\pi\underline{g} = 1$.
- Let H be a subgroup of G containing $\left\langle \underline{g} \right\rangle$. A conjugacy class γ of H *appears in* $\underline{g}$ if there is an $i \in \{1, \ldots, n\}$ for which $g_i \in \gamma$.
- Let H be a subgroup of G containing $\left\langle \underline{g} \right\rangle$, and let c be a union of conjugacy classes of H such that $g_1, \ldots, g_n \in c$. We denote by D_H^* the set of all conjugacy classes of H contained in c. The *(H, c)-multidiscriminant of* $\underline{g}$ is the map $\mu_{H,c}(\underline{g})\colon D_H^* \to \mathbb{Z}$ mapping a class $\gamma \in D_H^*$ to the number of times it appears in $\underline{g}$, i.e.:

$$\mu_{H,c}(\underline{g})(\gamma) \coloneqq \left| \left\{ i \in \{1, \ldots, n\} \;\middle|\; g_i \in \gamma \right\} \right|.$$

- If $\underline{g}' = (g'_1, \dots, g'_{n'})$ is another tuple of elements of G, the *concatenation* of $\underline{g}$ and $\underline{g}'$ is the tuple

$$\underline{gg'} \coloneqq (g_1, \dots, g_n, g'_1, \dots, g'_{n'}).$$

Note that $\deg(\underline{gg'}) = n + n' = \deg(\underline{g}) + \deg(\underline{g}')$, $\left\langle \underline{gg'} \right\rangle = \left\langle \underline{g}, \underline{g}' \right\rangle$, $\pi(\underline{gg'}) = (\pi\underline{g})(\pi\underline{g}')$, and $\mu_{H,c}(\underline{gg'}) = \mu_{H,c}(\underline{g}) + \mu_{H,c}(\underline{g}')$ if H contains $\left\langle \underline{g}, \underline{g}' \right\rangle$ and $g_1, \dots, g_n, g'_1, \dots, g'_{n'} \in c$.

2.1.3. Schemes. Let L be a field. In this article, *L-schemes* are schemes equipped with a separated morphism to $\mathrm{Spec}(L)$. Let $L'|L$ be a field extension, and let X be an L-scheme of finite type. We denote by $X_{L'}$ the L'-scheme $X \underset{\mathrm{Spec}(L)}{\times} \mathrm{Spec}(L')$ obtained by extending the scalars. The set $X(L')$ of L'-points of X is the set of morphisms of L-schemes from $\mathrm{Spec}(L')$ to X, or equivalently morphisms of L'-schemes from $\mathrm{Spec}(L')$ to $X_{L'}$. Any L-point $x \in X(L)$ induces an L'-point by composition with the map $\mathrm{Spec}(L') \to \mathrm{Spec}(L)$. An L'-point $x \in X(L')$ is *L-rational* if it is induced in that way by an L-point $x' \in X(L)$, and the point x' is then called an *L-model* of the point x. Similarly, an L'-subscheme Y of $X_{L'}$ is *defined over L* if there is an L-subscheme Y' of X such that the L'-subscheme $Y'_{L'} = Y \underset{X}{\times} X_{L'}$ of $X_{L'}$ coincides with Y, and then Y' is an *L-model* of Y.

2.2. Presentation of the main objects

In this subsection, we introduce configuration spaces (Paragraph 2.2.1), G-covers (Paragraphs 2.2.2 and 2.2.5) and Hurwitz spaces (Paragraphs 2.2.3 and 2.2.6) in both topological and algebraic settings, and the links between the two are explained. We also recall the combinatorial description of G-covers and of connected components of Hurwitz spaces (Paragraphs 2.2.3 and 2.2.4).

2.2.1. Configurations and braid groups. A *configuration* $\underline{t} = \{t_1, \dots, t_n\}$ is an unordered list of n distinct complex numbers. Configurations form a space $\mathrm{Conf}_n(\mathbb{C})$, whose topology is inherited from the standard topology on $\mathbb{C}^n$ after removing tuples with nondistinct coordinates, and quotienting out by the (free) action of $\mathfrak{S}_n$. The fundamental group of $\mathrm{Conf}_n(\mathbb{C})$ is the *Artin braid group* B_n, generated by the elementary braids $\sigma_1, \dots, \sigma_{n-1}$ subject to the following generating set of relations:

- $\sigma_i\sigma_j = \sigma_j\sigma_i$ for all $i, j \in \{1, \dots, n-1\}$ satisfying $|i - j| > 1$;
- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for all $i \in \{1, \dots, n-2\}$.

A configuration $\underline{t} \in \mathrm{Conf}_n(\mathbb{C})$ is *defined over K* if the elements $t_1, \dots, t_n$ are all algebraic and are permuted by the Galois action of $\mathrm{Gal}(\overline{\mathbb{Q}}|K)$, i.e., if the polynomial $\prod(X - t_i)$ belongs to $K[X]$. We denote by $\mathrm{Conf}_n(K)$ the set of configurations of $\mathrm{Conf}_n(\mathbb{C})$ defined over K.

The configuration space has a scheme counterpart. Indeed, fixing a configuration $\underline{t}$ amounts to fixing the squarefree monic polynomial $(X - t_1)\cdots(X - t_n)$ of degree n. Parametrizing these polynomials by their coefficients $1, a_1, \dots, a_n$ instead of their roots $t_1, \dots, t_n$, we obtain an open subvariety Conf_n of $\mathbb{A}^n$ by removing the closed subset defined by the vanishing of the discriminant of $X^n + a_1X^{n-1} + \ldots + a_{n-1}X + a_n$. The K-points of Conf_n correspond bijectively to the elements of $\mathrm{Conf}_n(K)$, and the set of its $\mathbb{C}$-points, equipped with the analytic topology, is homeomorphic to $\mathrm{Conf}_n(\mathbb{C})$, making the notation unambiguous.

2.2.2. Topological G-covers and tuples. In this article, the word "cover" refers to branched G-covers of the projective line. Let $\underline{t} \in \mathrm{Conf}_n(\mathbb{C})$ be a configuration. We write $\mathbb{P}^1(\mathbb{C}) \setminus \underline{t}$ for $\mathbb{P}^1(\mathbb{C}) \setminus \{t_1, \dots, t_n\}$. *Topological G-covers branched at $\underline{t}$* are covering maps $p \colon X \to \mathbb{P}^1(\mathbb{C}) \setminus \underline{t}$ (of degree $|G|$) equipped with a group homomorphism $G \to \mathrm{Aut}(p)$ inducing a free transitive G-action on each fiber. We do not assume that these covers are connected. A connected G-cover is a Galois covers whose automorphism group is isomorphic to G (more precisely, it is given with an isomorphism).

Remark 2.2. Note that we allow trivial ramification at the "branch points", even if the G-cover extends into a G-cover with fewer branch points. Therefore, "branched at $\underline{t}$" actually means "unramified outside $\underline{t}$".

A *marked G-cover* (branched at $\underline{t}$) is a G-cover $p\colon X \to \mathbb{P}^1(\mathbb{C}) \setminus \underline{t}$ equipped with a marked point $\star \in p^{-1}(\infty)$. If γ is a loop in $\mathbb{P}^1(\mathbb{C}) \setminus \underline{t}$ based at ∞, it lifts into a unique path in X starting at $\star$, whose endpoint belongs to $p^{-1}(\infty)$ and hence can be written as $\varphi([\gamma]).\star$ for some uniquely defined $\varphi([\gamma]) \in G$, depending only on the homotopy class $[\gamma]$ of the loop γ. Via this construction, every marked G-cover $(p, \star)$ branched at $\underline{t}$ induces a group homomorphism $\varphi\colon \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \underline{t}, \infty) \to G$. In fact, this leads to a bijection (analogous to [Sza09, Theorem 2.3.4])

$$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{marked } G\text{-covers of } \mathbb{P}^1(\mathbb{C}) \setminus \underline{t} \end{array} \right\} \xleftrightarrow{\sim} \mathrm{Hom}\Big(\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \underline{t},\, \infty),\, G\Big).$$

(For unmarked G-covers, one should instead consider G-conjugacy classes of homomorphisms.)

Choose a topological bouquet $(\gamma_1, \ldots, \gamma_n)$, as defined in [DE06, Paragraph 1.1]: this is a list of generators of $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \underline{t},\, \infty)$ where γ_i is the homotopy class of a loop which rotates once counterclockwise around the point t_i and becomes homotopically trivial when the point t_i is added back in, and the relations between these generators are generated by the single relation $\gamma_1 \cdots \gamma_n = 1$. Identifying the group homomorphism φ with the tuple $(\varphi(\gamma_1), \ldots, \varphi(\gamma_n))$ refines the bijection above:

$$\mathrm{Hom}\Big(\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \underline{t},\, \infty),\, G\Big) \xleftrightarrow{\sim} \left\{ \underline{g} \in G^n \,\middle|\, \pi\underline{g} = 1 \right\}. \qquad \text{(Recall that } \pi\underline{g} = g_1 \cdots g_n.\text{)}$$

If $(p, \star)$ is a marked G-cover, the corresponding n-tuple $(g_1, \ldots, g_n)$ is its *branch cycle description.* The *monodromy group* of $(p, \star)$, i.e., the automorphism group of the covering map restricted to the connected component of the marked point $\star$, is the subgroup $\left\langle \underline{g} \right\rangle = \langle g_1, \ldots, g_n \rangle$ of G. Finally, the marked G-cover $(p, \star)$ is connected if and only if $g_1, \ldots, g_n$ generate G, i.e., if its monodromy group $\left\langle \underline{g} \right\rangle$ is all of G.

Remark 2.3. We include non-connected G-covers, whose monodromy groups are proper subgroups of G, because we are interested in patching-like results. Typically, we want to construct components with monodromy group G by gluing components with smaller monodromy groups. If we do not take this phenomenon into account, the answer to Question 1.1 is "yes": the concatenation of two components defined over K of connected G-covers is always defined over K by Theorem 1.2 (i). In [Cau12], a different but equivalent choice is made: instead of considering components of marked non-connected G-covers, Cau considers components of unmarked connected H-covers where H is a subgroup of G. The links between these two approaches are discussed in Paragraph 2.3.4.

2.2.3. Topological Hurwitz spaces and their components. Unless specified otherwise, Hurwitz spaces in this article are moduli spaces of *marked* G-covers, connected or not. We denote by $\mathrm{Hur}^*(G, n)$ the topological Hurwitz space of marked G-covers with n branch points. That space admits a covering map to $\mathrm{Conf}_n(\mathbb{C})$ for which the fiber above a configuration $\underline{t} \in \mathrm{Conf}_n(\mathbb{C})$ consists of isomorphism classes of marked G-covers of $\mathbb{P}^1(\mathbb{C}) \setminus \underline{t}$ (cf. [Seg23, Definition 3.2.4] for a definition).

Classically, the Artin braid group B_n acts on the set of n-tuples of elements of G via the *Hurwitz action*, induced by the following formula:

$$\sigma_i.(g_1, \ldots, g_n) = (g_1, \ldots, g_{i-1}, g_{i+1}^{g_i}, g_i, \ldots, g_n). \tag{2.1}$$

We say that two tuples of elements of G are *braid equivalent* when they have the same size n and are in the same B_n-orbit. If $\underline{g}$ is a tuple of elements of G of size n, its *braid orbit* is its B_n-orbit. The following properties of the Hurwitz action are well-known, and their proofs are relatively straightforward (proofs can be found with this notation in [Seg23, Proposition 3.3.8, Proposition 3.3.11], or with different notation in [Cau12]):

Proposition 2.4. *The Hurwitz action satisfies the following properties:*

(i) The size, product, group, multidiscriminant of a tuple depend only on its braid orbit.

(ii) For any tuples $\underline{g}$ and $\underline{g}'$, the braid orbit of the concatenated tuple $\underline{g}\underline{g}'$ depends only on the braid orbits of $\underline{g}$ and $\underline{g}'$.

(iii) For any product-one tuples $\underline{g}$ and $\underline{g}'$, the concatenated tuples $\underline{g}\underline{g}'$ and $\underline{g}'\underline{g}$ are braid equivalent.

(iv) For any product-one tuples $\underline{g}_1, \underline{g}_2, \underline{g}_3$ and any $\gamma \in \left\langle \underline{g}_1, \underline{g}_3 \right\rangle \cup \left\langle \underline{g}_2 \right\rangle$, the tuple $\underline{g}_1\underline{g}_2\underline{g}_3$ is braid equivalent to $\underline{g}_1\underline{g}_2^{\gamma}\underline{g}_3$.

Proposition 2.4 (iv) is used in later proofs, notably that of Theorem 3.3. We often apply the special case where $\underline{g}_1$ and $\underline{g}_3$ are both empty: if $\pi\underline{g} = 1$ and $\gamma \in \left\langle \underline{g} \right\rangle$, then $\underline{g}$ and $\underline{g}^{\gamma}$ are braid equivalent.

A consequence of Proposition 2.4 (i) is that the braid orbit of a product-one tuple exclusively contains product-one tuples. The following proposition is classical (cf. [Seg23, Theorem 3.3.7 (ii)]):

Proposition 2.5. *The set of connected components of* $\mathrm{Hur}^*(G, n)$ *is in bijection with the set of braid orbits of product-one tuples* $\underline{g} = (g_1, \ldots, g_n) \in G^n$.

2.2.4. The monoid of components. By Proposition 2.5, the (graded) set of connected components of the space $\bigsqcup_{n\geq 0} \mathrm{Hur}^*(G, n)$ is in bijection with the (graded) set of braid orbits of product-one tuples:

$$\mathrm{Comp}(G) \coloneqq \bigsqcup_{n\geq 0} \left(\left\{ \underline{g} \in G^n \;\middle|\; \pi\underline{g} = 1 \right\} / \mathrm{B}_n \right).$$

We call the elements of $\mathrm{Comp}(G)$ *components*. The *monoid of components* $\mathrm{Comp}(G)$ is the graded monoid obtained by equipping this set with the grading given by the size n of a tuple, and with the product induced by concatenation, which is well-defined by Proposition 2.4 (ii), and is commutative by Proposition 2.4 (iii). The identity element of $\mathrm{Comp}(G)$ (of degree 0) is the orbit of the empty tuple, corresponding to the connected component containing the trivial G-cover. (Note that $\mathrm{Comp}(G)$ also has a single element of degree 1, the orbit of the tuple (1), corresponding to the trivial G-cover when it is seen as having one "actually unramified branch point", cf. Remark 2.2.)

Let H be a subgroup of G and c be a union of conjugacy classes of H. By Proposition 2.4 (i), there is a well-defined submonoid $\mathrm{Comp}(H, c) \subseteq \mathrm{Comp}(G)$ whose elements are braid orbits of tuples of elements of c, and the notations $\deg(x)$, $\langle x \rangle$, $\mu_{H,c}(x)$ from Paragraph 2.1.2 meaningfully extend to elements $x \in \mathrm{Comp}(H, c)$. We say that a conjugacy class $\gamma \subseteq c$ of H is a *monodromy class* of $x \in \mathrm{Comp}(H, c)$ if $\mu_{H,c}(x)(\gamma) \geq 1$ (i.e., in any tuple representing x, there is an element of γ).

2.2.5. Algebraic k-G-covers and Riemann's existence theorem. Let k be a field whose characteristic does not divide $|G|$. For us, an *algebraic cover* of $\mathbb{P}^1_k$ will refer to a finite flat generically étale k-morphism from a smooth projective curve Y over k (not assumed to be irreducible) to $\mathbb{P}^1_k$, unramified above the point at infinity. The $\overline{k}$-points of $\mathbb{P}^1$ at which an algebraic cover of $\mathbb{P}^1_k$ is ramified form a finite configuration $\underline{t} \in \mathrm{Conf}_n(k)$, for some n.

A *k-G-cover* is an algebraic cover $p\colon Y \to \mathbb{P}^1_k$ (of degree $|G|$) equipped with a group homomorphism from G to the group $\mathrm{Aut}(p)$ of k-automorphisms of the cover, inducing free and transitive G-actions on the set of geometric points of each unramified fiber. A k-G-cover *equipped with a marked k-point* is a k-G-cover $Y \to \mathbb{P}^1_k$ equipped with a k-point of Y in the unramified fiber above ∞.

If $p\colon Y \to \mathbb{P}^1_k$ is a k-G-cover with Y irreducible, then the induced extension $k(Y)|k(T)$ of function fields is Galois with Galois group G. This leads to an equivalence between the categories of Galois field extensions of $k(T)$ with Galois group G, and of k-G-covers (see [Har83, Corollary I.6.12] or [Stacks, Theorem 0BY1]). If Y is geometrically irreducible, the extension is also *regular*, i.e., $k(Y) \cap \overline{k} = k$.

Riemann's existence theorem (cf. [Sza09, Theorem 3.3.3 and Corollary 3.3.12]) implies that the category of $\mathbb{C}$-G-covers (resp. of $\overline{\mathbb{Q}}$-G-covers, cf. [Sza09, Theorem 4.6.10]) branched at some configuration $\underline{t} \in \mathrm{Conf}_n(\mathbb{C})$ (resp. at $\underline{t} \in \mathrm{Conf}_n(\overline{\mathbb{Q}})$) is equivalent to that of topological G-covers with the same branch points. Hence, we identify topological G-covers and k-G-covers over an algebraically closed field k of characteristic zero quite freely.

Assume now that $k \subseteq \overline{\mathbb{Q}}$. Since topological G-covers are well-understood, we look for *G-covers defined over k* instead of k-G-covers: a $\overline{\mathbb{Q}}$-G-cover $p\colon Y \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ is *defined over k* if it is isomorphic to the extension of scalars (to $\overline{\mathbb{Q}}$) of a k-G-cover $p_k\colon Y' \to \mathbb{P}^1_k$. In that case, we say that p_k is a *k-model* of the G-cover p. These notions will be defined again differently in Paragraph 2.3.2.

2.2.6. Hurwitz schemes. We denote by $\mathcal{H}^*(G,n)$ the Hurwitz space parametrizing *marked* G-covers of $(\mathbb{P}^1,\infty)$ whose branch locus has degree n, and which are unramified above ∞. Via the branch point map, the $\mathbb{Q}$-scheme $\mathcal{H}^*(G,n)$ is an étale cover of Conf_n. The K-points of $\mathcal{H}^*(G,n)$ correspond to algebraic K-G-covers branched at some configuration $\underline{t} \in \mathrm{Conf}_n(K)$, equipped with a marked K-point. The existence of this moduli space follows from [Ems95, Théorème 3] (see also [Kan24a, Kan24b]). The set of $\mathbb{C}$-points of $\mathcal{H}^*(G,n)$, equipped with the analytic topology, is homeomorphic to the space $\mathrm{Hur}^*(G,n)$ of Paragraph 2.2.3. The geometrically irreducible components of $\mathcal{H}^*(G,n)$ are in one-to-one correspondence with the connected components of $\mathrm{Hur}^*(G,n)$, and consequently with the B_n-orbits of product-one n-tuples of elements of G (i.e., the elements of degree n of $\mathrm{Comp}(G)$) by Proposition 2.5. Hence, we freely identify these various notions of "components".

The Hurwitz space $\mathcal{H}(G,n)$ parametrizing branched *unmarked* G-covers of $\mathbb{P}^1$ (which is a coarse moduli space) is only used in Paragraph 2.3.4, where we relate the fields of definition of components of $\mathcal{H}^*(G,n)$ to the more classical question of the fields of definition of components of $\mathcal{H}(G,n)$. That space is the quotient scheme $\mathcal{H}^*(G,n)/G$, where the action of G is interpreted in terms of tuples (cf. Paragraph 2.2.2) as the simultaneous conjugation of all elements of a tuple by the same element of G (corresponding to a change of marked point).

2.3. The Galois action

Fix a configuration $\underline{t} \in \mathrm{Conf}_n(K)$. In this subsection, we describe an action of $\Gamma_K = \mathrm{Gal}(\overline{\mathbb{Q}}|K)$ on the set of marked G-covers branched at $\underline{t}$, and on the set of the geometrically irreducible components of the corresponding Hurwitz space. We denote by $\pi_{1,\overline{\mathbb{Q}}}$ the étale fundamental group $\pi_1^{\text{ét}}(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \underline{t}, \infty)$ and by $\pi_{1,K}$ the étale fundamental group $\pi_1^{\text{ét}}(\mathbb{P}^1_K \setminus \underline{t}, \infty)$. The group $\pi_{1,\overline{\mathbb{Q}}}$ is isomorphic to the profinite completion of the topological fundamental group $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \underline{t}, \infty)$. Since G is finite, the universal property of profinite completions induces a bijection $\mathrm{Hom}\Big(\pi_{1,\overline{\mathbb{Q}}},\, G\Big) \overset{\sim}{\longleftrightarrow} \mathrm{Hom}\Big(\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \underline{t}, \infty),\, G\Big)$. Therefore, isomorphism classes of marked G-covers branched at $\underline{t}$ may and will be identified with continuous group homomorphisms $\pi_{1,\overline{\mathbb{Q}}} \to G$.

2.3.1. The Galois action on marked G-covers. Let $\overline{\mathbb{Q}(T)}$ be an algebraic closure of $\overline{\mathbb{Q}}(T)$. Let $\Omega_{\underline{t}}$ be the largest subfield of $\overline{\mathbb{Q}(T)}$ unramified outside $\underline{t}$. The chain of field extensions

$$\begin{array}{c} \Omega_{\underline{t}} \\ \pi_{1,\overline{\mathbb{Q}}} \Big| \\ \overline{\mathbb{Q}}(T) \\ \Gamma_K \Big| \\ K(T) \end{array} \quad \Bigg) \, \pi_{1,K}$$

induces the following short exact sequence of Galois groups:

$$1 \longrightarrow \pi_{1,\overline{\mathbb{Q}}} \longrightarrow \pi_{1,K} \longrightarrow \Gamma_K \longrightarrow 1. \tag{2.2}$$

The field $\Omega_{\underline{t}} \subseteq \overline{\mathbb{Q}(T)}$ embeds in the field of Puiseux series $\overline{\mathbb{Q}}(((1/T)^{1/\infty}))$, on which Γ_K acts coefficientwise. Since the configuration $\underline{t}$ is defined over K, the field $\Omega_{\underline{t}}$ is stable under the action of Γ_K. So, there is an action of Γ_K on $\Omega_{\underline{t}}$, trivial on $K(T)$. This defines a group homomorphism:

$$s\colon \Gamma_K \to \mathrm{Gal}(\Omega_{\underline{t}} \mid K(T)) \simeq \pi_{1,K}$$

associated with the choice of a base K-point (here, the point at infinity). The homomorphism s is a section of the short exact sequence of Equation (2.2):

$$1 \longrightarrow \pi_{1,\overline{\mathbb{Q}}} \longrightarrow \pi_{1,K} \underset{\curvearrowleft s}{\longrightarrow} \Gamma_K \longrightarrow 1.$$

Using the section s, we define an action of Γ_K on $\pi_{1,\overline{\mathbb{Q}}}$: an element $\sigma \in \Gamma_K$ acts on an element $\gamma \in \pi_{1,\overline{\mathbb{Q}}}$ by mapping it to $\sigma.\gamma := \gamma^{s(\sigma)}$, which belongs to $\pi_{1,\overline{\mathbb{Q}}}$ as $\pi_{1,\overline{\mathbb{Q}}}$ is normal in $\pi_{1,K}$. We then let $\sigma \in \Gamma_K$ act on an isomorphism class of marked G-covers, seen as a continuous group homomorphism $\varphi\colon \pi_{1,\overline{\mathbb{Q}}} \to G$, by mapping it to the continuous group homomorphism

$$\sigma.\varphi\colon \begin{cases} \pi_{1,\overline{\mathbb{Q}}} & \to \quad G \\ \gamma & \mapsto \quad \varphi(\sigma.\gamma) = \varphi(\gamma^{s(\sigma)}). \end{cases} \tag{2.3}$$

This action preserves the monodromy group of a marked G-cover branched at $\underline{t}$. Moreover, we have $\sigma.(\varphi^g) = (\sigma.\varphi)^g$ for all $g \in G$. In particular, there is a well-defined action of Γ_K on isomorphism classes of unmarked G-covers branched at $\underline{t}$, identified with conjugacy classes of continuous group homomorphisms $\pi_{1,\overline{\mathbb{Q}}} \to G$. (In terms of morphisms of curves, this action corresponds to pulling back an unmarked G-cover $Y \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ along the automorphism of $\mathbb{P}^1_{\overline{\mathbb{Q}}}$ induced by σ.)

2.3.2. Fields of definition of covers. Consider an isomorphism class of marked branched G-covers, seen as a continuous group homomorphism $\varphi\colon \pi_{1,\overline{\mathbb{Q}}} \to G$.

Definition 2.6. The marked G-cover associated to φ is *defined over* K if $\sigma.\varphi = \varphi$ for all $\sigma \in \Gamma_K$.

The equivalence with the definition given in Paragraph 2.2.5 (marked G-covers defined over K are obtained by extension of scalars of K-G-covers equipped with a marked K-point)[1] follows from the properties of étale fundamental groups and from the following proposition:

Proposition 2.7. *The marked cover associated to $\varphi\colon \pi_{1,\overline{\mathbb{Q}}} \to G$ is defined over K if and only if φ extends into a continuous group homomorphism $\widetilde{\varphi}\colon \pi_{1,K} \to G$ which is trivial on* $\mathrm{Im}(s)$.

(The triviality of $\widetilde{\varphi}$ on $\mathrm{Im}(s)$ means that the marked point above ∞ is K-rational, since the group homomorphism $\widetilde{\varphi} \circ s\colon \Gamma_K \to G$ describes the action of Γ_K on the marked point.)

Proof. Give names to the homomorphisms in the short exact sequence of Equation (2.2):

$$\pi_{1,\overline{\mathbb{Q}}} \overset{\iota}{\hookrightarrow} \pi_{1,K} \overset{w}{\twoheadrightarrow} \Gamma_K$$

and remember that $s\colon \Gamma_K \to \pi_{1,K}$ is a section of w, i.e., that $w \circ s = \mathrm{id}_{\pi_{1,K}}$.

($\Leftarrow$) Assume that there is a group homomorphism $\widetilde{\varphi}\colon \pi_{1,K} \to G$ such that $\varphi = \widetilde{\varphi} \circ \iota$ and $\widetilde{\varphi} \circ s = 1$. For any $\gamma \in \pi_{1,\overline{\mathbb{Q}}}$ and $\sigma \in \Gamma_K$, we have

$$(\sigma.\varphi)(\gamma) = \varphi(\gamma^{s(\sigma)}) = \widetilde{\varphi}(\gamma^{s(\sigma)}) = \widetilde{\varphi}(\gamma)^{\widetilde{\varphi}(s(\sigma))} = \widetilde{\varphi}(\gamma) = \varphi(\gamma).$$

[1]The claim is that there is a unique K-model whose marked point is K-rational. However, there may also be K-models for which this is not the case: in fact, for each continuous group homomorphism $\mu\colon \Gamma_K \to \mathrm{Cent}_G(\mathrm{Im}(\varphi))$, there is exactly one K-model such that the Galois action on the marked point is given by μ.

(⇒) Assume that φ is defined over K. For any $\gamma \in \pi_{1,K}$, we have

$$w(\gamma s(w(\gamma))^{-1}) = w(\gamma)\ (w \circ s \circ w)(\gamma)^{-1} = w(\gamma)w(\gamma)^{-1} = 1,$$

which implies that $\gamma s(w(\gamma))^{-1} \in \pi_{1,\overline{\mathbb{Q}}}$ by exactness of Equation (2.2). Define the map

$$\widetilde{\varphi}\colon \begin{cases} \pi_{1,K} & \to & G \\ \gamma & \mapsto & \varphi(\gamma s(w(\gamma))^{-1}). \end{cases}$$

If $\gamma \in \pi_{1,\overline{\mathbb{Q}}}$, we have $w(\gamma) = 1$ and then $\widetilde{\varphi}(\gamma) = \varphi(\gamma)$. If $\sigma \in \Gamma_K$, then $s(w(s(\sigma))) = s(\sigma)$, so $\widetilde{\varphi}(s(\sigma)) = \varphi(s(\sigma)s(\sigma)^{-1}) = \varphi(1) = 1$. We have shown that $\varphi = \widetilde{\varphi} \circ \iota$ and $\widetilde{\varphi} \circ s = 1$. It remains only to check that $\widetilde{\varphi}$ is a group homomorphism. For any $x, y \in \pi_{1,K}$, we have

$$\begin{aligned}
\widetilde{\varphi}(x)\widetilde{\varphi}(y) &= \varphi(xs(w(x))^{-1})\,\varphi(ys(w(y))^{-1}) && \text{by definition of } \widetilde{\varphi} \\
&= \varphi(xs(w(x))^{-1})\,(w(x).\varphi)(ys(w(y))^{-1}) && \text{as } \varphi \text{ is defined over } K \\
&= \varphi(xs(w(x))^{-1})\,\varphi(s(w(x))ys(w(y))^{-1}s(w(x))^{-1}) && \text{by definition of the } \Gamma_K\text{-action} \\
&= \varphi(xys(w(xy))^{-1}) = \widetilde{\varphi}(xy) && \text{by definition of } \widetilde{\varphi}. \quad \square
\end{aligned}$$

Remark 2.8. A naive analogue of Definition 2.6 for unmarked G-covers would be to require that a conjugacy class $[\varphi]$ of continuous group homomorphisms $\pi_{1,\overline{\mathbb{Q}}} \to G$ be invariant under the Galois action, i.e., that for all $\sigma \in \Gamma_K$ we have $\sigma.\varphi = \rho(\sigma)^{-1}\varphi\rho(\sigma)$ for some $\rho(\sigma) \in G$. However, to adapt the proof of Proposition 2.7, we need ρ to be a homomorphism, but this is not always possible.[2] This is the main reason why we work with marked covers in this paper. For additional details, see [DD97].

2.3.3. The Galois action on components and fields of definition. The Galois action on marked G-covers (defined above in the case where the branch locus is defined over K) comes from the natural action of Γ_K on the Hurwitz scheme of marked G-covers. As such, this action maps geometrically irreducible components to geometrically irreducible components,[3] i.e., there is a well-defined Γ_K-action on the graded set $\mathrm{Comp}(G)$, preserving both the degree (number of branch points) and the monodromy group of components. We do not claim, however, that this action is compatible with the monoid structure of $\mathrm{Comp}(G)$: this is precisely the difficulty of Question 1.1. We use this action to define the notion of field of definition of components:

Definition 2.9. A component $m \in \mathrm{Comp}(G)$ is *defined over* K if for all $\sigma \in \Gamma_K$ we have $\sigma.m = m$.

To see that this matches the definition of Paragraph 2.1.3, see [Stacks, Lemma 038D].

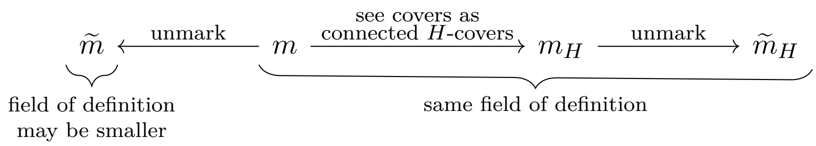
2.3.4. Comparison between the marked and the unmarked cases. Let $m \in \mathrm{Comp}(G)$ be a component of $\mathcal{H}^*(G,n)_{\overline{\mathbb{Q}}}$, and let $\widetilde{m}$ be the component of $\mathcal{H}(G,n)_{\overline{\mathbb{Q}}}$ obtained by forgetting the marked points of the G-covers in m. The component m is defined over K when $\sigma.m = m$ for all $\sigma \in \Gamma_K$. The component $\widetilde{m}$ is defined over K when, for all $\sigma \in \Gamma_K$, there is a $\gamma(\sigma) \in G$ such that $\sigma.m = m^{\gamma(\sigma)}$. In general, the latter property is weaker than the former. However, if $\langle m \rangle = G$ (the G-covers in m are connected), then $m^\gamma = m$ for all $\gamma \in G$ by Proposition 2.4 (iv), so there is no difference between m and $\widetilde{m}$ being defined over K.

Assume now that $H := \langle m \rangle$ is a proper subgroup of G. Define a component m_H of $\mathcal{H}^*(H,n)_{\overline{\mathbb{Q}}}$ by isolating the connected component containing the marked point of each cover in m, thereby turning

[2] Assume that φ is surjective. Then, there is at most one possible value for $\rho(\sigma) \bmod Z(G)$, so the map $\overline{\rho}\colon \Gamma_K \to G/Z(G)$ induced by ρ is a group homomorphism. The exact sequence $H^1(\Gamma_K, G) \to H^1(\Gamma_K, G/Z(G)) \to H^2(\Gamma_K, Z(G))$ of pointed sets shows that the obstruction to lifting $\overline{\rho}$ into a group homomorphism $\Gamma_K \to G$ lies in $H^2(\Gamma_K, Z(G))$. In particular, if $H^2(\Gamma_K, Z(G))$ is trivial (e.g., if G is centerless), then there is no obstruction.

[3] Another way of seeing this is to notice that both the B_n-action and the Γ_K-action on marked G-covers come from an action of the "arithmetic braid group" $\pi_1^{\text{ét}}\big((\mathrm{Conf}_n)_K, \underline{t}\big) \simeq \widehat{\mathrm{B}}_n \rtimes \Gamma_K$ on $\pi_{1,\overline{\mathbb{Q}}}$, where $\widehat{\mathrm{B}}_n = \pi_1^{\text{ét}}\big((\mathrm{Conf}_n)_{\overline{\mathbb{Q}}}, \underline{t}\big)$ is the profinite completion of B_n.

it into a connected H-cover. Let also $\widetilde{m}_H$ be the component of $\mathcal{H}(H,n)_{\overline{\mathbb{Q}}}$ obtained by forgetting the marked points of the connected H-covers in m_H. By the connected case (previous paragraph), the fields of definition of m_H and $\widetilde{m}_H$ are the same. Moreover, since the definition of $\sigma.m$ is independent from the ambient group, m is defined over K if and only if m_H is defined over K. However, $\widetilde{m}$ may have a field of definition smaller than that of m (an example is given in [Seg23, Exemple 7.2.12]). The situation is summarized by the following diagram:



Therefore, studying the fields of definitions of components of marked G-covers with monodromy group H (as we do) is equivalent to studying the fields of definitions of components of geometrically connected unmarked H-covers, as is done in [Cau12]. We opt for the former approach because it allows for a unified treatment of all components and leads to a simpler algebraic structure (the monoid of components). As a final remark, we note that there are still ways to relate the fields of definition of $\widetilde{m}$ and m:

Lemma 2.10. *If $\widetilde{m}$ is defined over K and H is either self-normalizing in G or has no outer automorphisms, then m is defined over K.*

Proof. Consider some $\sigma \in \Gamma_K$. By looking at the monodromy groups, the equality $\sigma.m = m^\gamma$ implies that $H = H^\gamma$, so conjugation by γ defines an automorphism of H.

- If H is self-normalizing, this implies $\gamma \in H$.
- If H has no outer automorphisms, the automorphism of H induced by conjugation by γ has to be inner, so there is a $\gamma' \in H$ such that $h^\gamma = h^{\gamma'}$ for all $h \in H$, and then $m^\gamma = m^{\gamma'}$.

In both cases, Proposition 2.4 (iv) implies that $\sigma.m = m^\gamma = m$. Therefore, m is defined over K. ☐

2.3.5. The branch cycle lemma. The effect of the Galois action on multidiscriminants is well-known. Let $\underline{t} \in \mathrm{Conf}_n(K)$, let $(\gamma_1, \ldots, \gamma_n) \in \pi_{1,\overline{\mathbb{Q}}}^n$ be a bouquet associated to $\underline{t}$, and consider a marked G-cover branched at $\underline{t}$, seen as a continuous group homomorphism $\varphi\colon \pi_{1,\overline{\mathbb{Q}}} \to G$. For the following classical result, we refer to [Fri77]/[Sza09, Remark 4.8.8] (or to [Cau12, Lemme 2.2] for a statement closer to ours):

Lemma 2.11. *For every $\sigma \in \Gamma_K$, the element $(\sigma.\varphi)(\gamma_i)$ is conjugate to $\varphi(\gamma_i)^{\chi(\sigma)^{-1}}$.*

We are going to interpret Lemma 2.11 in terms of multidiscriminants (in Corollary 2.13). Let $\underline{g} = (\varphi(\gamma_1), \ldots, \varphi(\gamma_n))$ be the tuple associated to φ, and $\sigma.\underline{g} = (\varphi(\sigma.\gamma_1), \ldots, \varphi(\sigma.\gamma_n))$ be the tuple associated to $\sigma.\varphi$ for any $\sigma \in \Gamma_K$. Let H be a subgroup of G containing $\left\langle \underline{g} \right\rangle$ and c be a K-rational (cf. Definition 2.1) union of conjugacy classes of H containing $g_1, \ldots, g_n$. Let D^* be the set of conjugacy classes of H contained in c, and let p_σ be the map $D^* \to D^*$ induced by the $\chi(\sigma)$-th power, for any $\sigma \in \Gamma_K$. Let $x \in \mathrm{Comp}(G)$ be the component represented by the tuple $\underline{g}$. Recall from Subsection 2.1 that the (H,c)-multidiscriminant $\mu_{H,c}(x)$ of x is the map that counts the appearances in $\underline{g}$ of each conjugacy class in D^*.

Definition 2.12. We say that x *has a K-rational (H,c)-multidiscriminant* if $\mu_{H,c}(x) = \mu_{H,c}(x) \circ p_\sigma$ for all $\sigma \in \Gamma_K$, i.e., if each conjugacy class of H appears equally often in $\underline{g}$ as its $\chi(\sigma)$-th power.

(Note that a concatenation of components with K-rational multidiscriminants also has a K-rational multidiscriminant.)

Corollary 2.13.

(i) For all $\sigma \in \Gamma_K$, we have $\mu_{H,c}(\sigma.x) = \mu_{H,c}(x) \circ p_\sigma$.

(ii) If x is defined over K, then x has a K-rational (H, c)-multidiscriminant.

(iii) If x has a K-rational (H, c)-multidiscriminant and H is abelian, then x is defined over K.

Point (ii) gives an easily checked necessary condition for a component to be defined over K, and point (iii) says that this condition is also sufficient when the group is abelian.

Proof.

(i) Let $\gamma \in D^*$. Then, $\mu_{H,c}(\sigma.x)(\gamma)$ is the number of appearances of γ in $\sigma.\underline{g}$. By Lemma 2.11, this is also the number of appearances of $\gamma^{\chi(\sigma)}$ in $\underline{g}$, which is $\mu_{H,c}(x)(\gamma^{\chi(\sigma)}) = (\mu_{H,c}(x) \circ p_\sigma)(\gamma)$.

(ii) Since x is defined over K, we have $\sigma.x = x$ for all $\sigma \in \Gamma_K$. By point (i), this implies $\mu_{H,c}(x) = \mu_{H,c}(x) \circ p_\sigma$, i.e., x has a K-rational (H, c)-multidiscriminant.

(iii) Since H is abelian, conjugacy classes of H and elements of H are "the same", and components are just unordered product-one tuples of elements of H (braid groups act by permutation). Hence, components are uniquely determined by their (H, c)-multidiscriminants.

By hypothesis, x has a K-rational (H, c)-multidiscriminant, so for all $\sigma \in \Gamma_K$ we have $\mu_{H,c}(x) = \mu_{H,c}(x) \circ p_\sigma$, which is $\mu_{H,c}(\sigma.x)$ by (i). Since x and $\sigma.x$ have the same (H, c)-multidiscriminant, they are equal. □

Example 2.14. Assume that G is abelian. Then, Corollary 2.13 (iii) allows one to determine the field of definition of components. A consequence is that the answer to Question 1.1 is "yes" in this case. For example, the component represented by $\underline{g} \in G^n$ is defined over $\mathbb{Q}$ if and only if every $g \in G$ appears as many times in $\underline{g}$ as the elements g^k for k coprime with $\text{ord}(g)$. We give two examples:

- The component $(1, 1, 1) \in \text{Comp}(\mathbb{Z}/3\mathbb{Z})$ is not defined over $\mathbb{Q}$, because 1 does not appear as many times as -1.
- The component $(1, -1) \in \text{Comp}(\mathbb{Z}/n\mathbb{Z})$ is defined over $\mathbb{Q}$ for $n \in \{2, 3, 4, 6\}$, and not defined over $\mathbb{Q}$ for $n = 5$ or $n \geq 7$.

Example 2.15. It follows from [Seg23, Theorem 6.2.6] that components of $\mathfrak{S}_d$-covers whose monodromy elements are transpositions are entirely determined by their monodromy group H and their (H, H)-multidiscriminant. Since transpositions are involutions, all conjugacy classes involved are $\mathbb{Q}$-rational, therefore Lemma 2.11 implies that the action of $\text{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ preserves multidiscriminants. Since components of $\mathfrak{S}_d$-covers whose monodromy elements are transpositions are determined by their multidiscriminants, which are $\mathbb{Q}$-rational, these components are all defined over $\mathbb{Q}$.

3. The group-theoretic approach

In this section, we propose new applications of some ideas introduced in [Cau12], which we recall in Subsection 3.1. In Subsection 3.2, we prove Theorem 3.3, whose third point (for $n = 2$) corresponds to Theorem 1.2 (i): this result gives a group-theoretic condition ensuring that a product of components defined over K is defined over K. Applications are given in Subsection 3.3.

3.1. Cau's theorem

Let $x_1, \dots, x_n \in \mathrm{Comp}(G)$ be components, and let H be a subgroup of G containing $\langle x_1, \dots, x_n \rangle$. Following [Cau12], we define the following subsets of $\mathrm{Comp}(G)$:

$$\mathrm{ni}_H(x_1, \dots, x_n) = \left\{ \prod_{i=1}^n x_i^{\gamma_i} \;\middle|\; (\gamma_1, \dots, \gamma_n) \in H^n \right\}.$$

$$\mathrm{ni}_H^\natural(x_1, \dots, x_n) = \left\{ \prod_{i=1}^n x_i^{\gamma_i} \;\middle|\; \begin{array}{c} (\gamma_1, \dots, \gamma_n) \in H^n \\ \langle x_1^{\gamma_1} \cdots x_n^{\gamma_n} \rangle = \langle x_1 \cdots x_n \rangle \end{array} \right\}.$$

We have $x_1 \cdots x_n \in \mathrm{ni}_H^\natural(x_1, \dots, x_n) \subseteq \mathrm{ni}_H(x_1, \dots, x_n)$. When H is omitted in the notation, it is implied that $H = \langle x_1 \cdots x_n \rangle$.

In Cau's terminology, the list $(x_1, \dots, x_n)$ of elements of $\mathrm{Comp}(G)$ corresponds to a *degenerescence structure* Δ, and the elements of $\mathrm{ni}_H(\Delta)$ are the Δ-*components*. Cau gives criteria to recognize Δ-components based on the existence of a "Δ-admissible cover" on their boundary. This characterization is key for his proof of the following theorem, which is [Cau12, Théorème 3.2]:

Theorem 3.1. *The action of any $\sigma \in \Gamma_K$ induces a bijection $\mathrm{ni}_H(x_1, \dots, x_n) \xrightarrow{\sim} \mathrm{ni}_H(\sigma.x_1, \dots, \sigma.x_n)$, and the same holds if ni_H is replaced by $\mathrm{ni}_H^\natural$.*

(That Theorem 3.1 holds if ni_H is replaced with $\mathrm{ni}_H^\natural$ follows from the fact that the Galois action preserves monodromy groups.)

If X is a finite set of components and $\sigma \in \Gamma_K$, we write Theorem 3.1 under the form $\sigma.\mathrm{ni}(X) = \mathrm{ni}(\sigma.X)$, where $\sigma.X$ is a shorthand for $\{\sigma.x \mid x \in X\}$.

3.2. Permuting components

In [Cau12, Proposition 2.10] and [Cau16, Théorème 3.8], Cau applies Theorem 3.1 in situations where he proves $\mathrm{ni}(x_1, \dots, x_n) = \{x_1 \cdots x_n\}$. We introduce a weaker condition, and we show that it implies $\mathrm{ni}^\natural(x_1, \dots, x_n) = \{x_1 \cdots x_n\}$ (cf. Theorem 3.3 (i)):

Definition 3.2. Let $x_1, \dots, x_n \in \mathrm{Comp}(G)$ be components, let $H_i = \langle x_i \rangle$, and let $H = \langle H_1, \dots, H_n \rangle$. The family $(x_1, \dots, x_n)$ is *permuting* if, for all elements $\gamma_3, \gamma_4, \dots, \gamma_n \in H$ and for all $i \in \{2, \dots, n\}$, we have the following implication:

$$\text{if } \left\langle H_1, H_2, \dots, H_{i-1}, H_i, H_{i+1}^{\gamma_{i+1}}, \dots, H_n^{\gamma_n} \right\rangle = H,$$
$$\text{then } \left\langle H_1, H_2, \dots, H_{i-1}, H_{i+1}^{\gamma_{i+1}}, \dots, H_n^{\gamma_n} \right\rangle H_i = H.$$

In particular, two components x_1, x_2 are permuting if and only if the subgroups $H_1 = \langle x_1 \rangle$ and $H_2 = \langle x_2 \rangle$ of G satisfy $H_1 H_2 = \langle H_1, H_2 \rangle$ (such subgroups are sometimes called permuting subgroups, hence the terminology). For instance, this holds whenever H_1 or H_2 is normal in $\langle H_1, H_2 \rangle$, e.g., when $H_1 \supseteq H_2$, or when $H = H_1 \ltimes H_2$ (thus improving [Cau12, Théorème 3.5]). Note also that whether a family of components is permuting only depends on their monodromy groups, and hence this property is preserved by the Galois action.

We now prove Theorem 3.3, whose third point (for $n = 2$) is Theorem 1.2 (i):

Theorem 3.3. *Let $(x_1, \dots, x_n)$ be a permuting family of components. Then:*

(i) $\mathrm{ni}_H^\natural(x_1, \dots, x_n) = \{x_1 \cdots x_n\}$.

(ii) For all $\sigma \in \Gamma_K$, we have $\sigma.(x_1 \cdots x_n) = (\sigma.x_1) \cdots (\sigma.x_n)$.

(iii) If $x_1, \dots, x_n$ are defined over K, then $x_1 \cdots x_n$ is defined over K.

Proof.

(i) Let $\gamma_1, \dots, \gamma_n \in H$ be such that $\langle \prod x_i^{\gamma_i} \rangle = H$. We want to show that $x_1^{\gamma_1} \cdots x_n^{\gamma_n} = x_1 \cdots x_n$. We use Proposition 2.4 (iv) to reduce to the case $\gamma_1 = 1$ (conjugate $x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ by $\gamma_1^{-1} \in \langle \prod x_i^{\gamma_i} \rangle$, replacing γ_i by $\gamma_1^{-1}\gamma_i$). We then prove the following equality by decreasing induction on i (the case $i = n$ is tautological, and the case $i = 1$ is our goal):

$$x_1 \cdots x_n = x_1 x_2 \cdots x_{i-1} x_i x_{i+1}^{\gamma_{i+1}} \cdots x_n^{\gamma_n}.$$

Assume that this equality holds for a fixed $i \in \{2, \dots, n\}$. In particular, we have

$$H = \left\langle H_1, H_2, \dots, H_{i-1}, H_i, H_{i+1}^{\gamma_{i+1}}, \dots, H_n^{\gamma_n} \right\rangle.$$

Since $(x_1, \dots, x_n)$ is permuting, we can write

$$\gamma_i = \gamma_i^{(1)} \gamma_i^{(2)} \qquad \text{where } \gamma_i^{(1)} \in \left\langle H_1, H_2, \dots, H_{i-1}, H_{i+1}^{\gamma_{i+1}}, \dots, H_n^{\gamma_n} \right\rangle \text{ and } \gamma_i^{(2)} \in H_i.$$

Therefore:

$$\begin{aligned}
x_1 \cdots x_n &= x_1 x_2 \cdots x_{i-1} x_i x_{i+1}^{\gamma_{i+1}} \cdots x_n^{\gamma_n} && \text{by the induction hypothesis} \\
&= x_1 \cdots x_{i-1} x_i^{\gamma_i^{(2)}} x_{i+1}^{\gamma_{i+1}} \cdots x_n^{\gamma_n} && \text{by Proposition 2.4 (iv)} \\
&= x_1 \cdots x_{i-1} \left(x_i^{\gamma_i^{(2)}} \right)^{\gamma_i^{(1)}} x_{i+1}^{\gamma_{i+1}} \cdots x_n^{\gamma_n} && \text{by Proposition 2.4 (iv)} \\
&= x_1 \cdots x_{i-1} x_i^{\gamma_i} x_{i+1}^{\gamma_{i+1}} \cdots x_n^{\gamma_n}.
\end{aligned}$$

We conclude by induction.

(ii) Let $\sigma \in \Gamma_K$. By Theorem 3.1, the component $\sigma.(x_1 \cdots x_n)$ belongs to the set $\mathrm{ni}^\natural(\sigma.x_1, \dots, \sigma.x_n)$. By point (i) applied to the permuting components $\sigma.x_1, \dots, \sigma.x_n$, we have $\mathrm{ni}^\natural(\sigma.x_1, \dots, \sigma.x_n) = \{(\sigma.x_1) \cdots (\sigma.x_n)\}$, so $\sigma.(x_1 \cdots x_n) = (\sigma.x_1) \cdots (\sigma.x_n)$.

(iii) This follows immediately from point (ii). □

Remark 3.4. One can also deduce point (iii) from point (i) using Theorem 5.4 instead of Theorem 3.1.

3.3. Applications

We give applications of Theorem 3.3. We first describe the Γ_K-action on powers of components:

Corollary 3.5. *For all $x \in \mathrm{Comp}(G)$, for all $\sigma \in \Gamma_K$, and for all $n \geq 0$, we have $\sigma.(x^n) = (\sigma.x)^n$. In particular, if x is defined over K, then so is x^n.*

Proof. Apply Theorem 3.3 to the permuting family $(\underbrace{x, \dots, x}_{n})$. □

The next result, Corollary 3.6, is a variant of [Cau12, Corollaire 1.1/Corollaire 3.4]. Cau shows that the concatenation of all components with a given size is defined over $\mathbb{Q}$, whereas we restrict our attention to a single Galois orbit, leading to a smaller size for the product component:

Corollary 3.6. *Let $x \in \mathrm{Comp}(G)$ be a component, and let $\Gamma_K.x \coloneqq \{\sigma.x \mid \sigma \in \Gamma_K\}$ be its Galois orbit. Then, the following component is defined over K:*

$$\mathrm{N}_K(x) \coloneqq \prod_{x' \in \Gamma_K.x} x'.$$

Proof. Let $H = \langle x \rangle$. Since all components of the form $\sigma.x$ have group H, they form a permuting family and thus Theorem 3.3 (i) implies that

$$\mathrm{ni}^{\natural}_H(\Gamma_K.x) = \{\mathrm{N}_K(x)\}. \tag{3.1}$$

Consider an automorphism $\sigma \in \Gamma_K$. The action of σ permutes $\Gamma_K.x$. Finally:

$$\begin{aligned} \{\mathrm{N}_K(x)\} &= \mathrm{ni}^{\natural}_H(\Gamma_K.x) && \text{by Equation (3.1)} \\ &= \mathrm{ni}^{\natural}_H(\sigma.(\Gamma_K.x)) && \text{because } \sigma \text{ permutes } \Gamma_K.x \\ &= \sigma.\mathrm{ni}^{\natural}_H(\Gamma_K.x) && \text{by Theorem 3.1} \\ &= \{\sigma.\mathrm{N}_K(x)\} && \text{by Equation (3.1)}, \end{aligned}$$

and thus $\mathrm{N}_K(x)$ is defined over K. □

Corollary 3.7. *Let c be a K-rational union of conjugacy classes of G. Assume that c is* complete, *i.e., that no proper subgroup of G intersects every conjugacy class contained in c.*[4] *Then, the following component (which also appears in [EVW12, Paragraph 5.5]) is defined over K:*

$$V := \prod_{g \in c} (\underbrace{g, \dots, g}_{\mathrm{ord}(g)}).$$

Proof. Let $\sigma \in \Gamma_K$. For any $g \in c$, the group $\langle g \rangle$ is abelian and so $\sigma.(g, \dots, g) = (g^{\chi(\sigma^{-1})}, \dots, g^{\chi(\sigma^{-1})})$ by Corollary 2.13 (i). Since the profinite integer $\chi(\sigma^{-1})$ is invertible, the action of σ permutes the factors of V. In order to apply Theorem 3.3 (ii) to show that $\sigma.V = \prod_g (g^{\chi(\sigma^{-1})}, \dots, g^{\chi(\sigma^{-1})}) = V$, we verify that the family of components $(g, \dots, g)$ for $g \in c$ is permuting. Consider an element $g \in c$ and elements $\gamma_{g'} \in G$ for all $g' \in c \setminus \{g\}$ such that G is generated by g together with the elements $(g')^{\gamma_{g'}}$ for $g' \in c \setminus \{g\}$. We want to show that $\langle (g')^{\gamma_{g'}} \text{ for } g' \in c \setminus \{g\} \rangle \langle g \rangle = G$. There are two cases:

- If g is central in G, then this follows easily from, say, the fact that $\langle g \rangle$ is normal in G.
- If g is not central in G, then there is a $g' \in c \setminus \{g\}$ such that g and g' are conjugate. Therefore, $\langle (g')^{\gamma_{g'}} \text{ for } g' \in c \setminus \{g\} \rangle$ is a subgroup of G that intersects every conjugacy class contained in c, and thus it equals G by the completeness assumption. □

4. The lifting invariant approach

In this section, we apply the lifting invariant of [EVW12, Woo21] to Question 1.1, in order to prove Theorem 4.7 (which is Theorem 1.2 (ii)). We first recall known properties of this invariant (Subsection 4.1) and then give arithmetic applications (Subsections 4.2 and 4.3).

4.1. Presentation of the lifting invariant

In what follows, H is a subgroup of G and D^* is a set of conjugacy classes of H which together generate H. We let $c := \bigsqcup_{\gamma \in D^*} \gamma$, and we let $\mathrm{Comp}(H, c)$ be the submonoid of $\mathrm{Comp}(G)$ whose elements are braid orbits of product-one tuples of elements of c (i.e., components whose monodromy group is contained in H, and whose monodromy classes belongs to H).

[4] An example of a complete subset is $c = G \setminus \{1\}$ (this fact is sometimes called Jordan's lemma). Here is a proof: if H is a proper subgroup of G intersecting all conjugacy classes, then $G = \bigcup_{\gamma \in G} H^\gamma$, and H cannot be normal, so its normalizer $N_G(H)$ is a proper subgroup of G. But then, $G = \{1\} \sqcup \bigcup_{\gamma \in G/N_G(H)} (H \setminus \{1\})^\gamma$ has size at most $1 + \frac{|G|}{|N_G(H)|}(|H| - 1) \le 1 + 2(|H| - 1) = 2|H| - 1 < |G|$, which is absurd.

4.1.1. Definition and first properties. Let $U(H,c)$ be the group defined by the following presentation: generators are given by elements $[g]$ for each $g \in c$, subject to the relations $[g][h][g]^{-1} = [ghg^{-1}]$ for all $g, h \in c$.

Definition 4.1. The (H,c)*-lifting invariant* $\Pi_{H,c}(\underline{g})$ of a tuple $\underline{g} = (g_1, \dots, g_n) \in c^n$ is the element $[g_1] \cdots [g_n] \in U(H,c)$.

Proposition 4.2. *The (H,c)-lifting invariant of $\underline{g}$ depends only on the braid orbit of $\underline{g}$.*

Proof. The relation $[g][h][g]^{-1} = [ghg^{-1}]$ in $U(H,c)$ can be rewritten as $[g][h] = [h^g][g]$. Comparing with Equation (2.1), one notices that the (H,c)-lifting invariant is unchanged by elementary braids, which generate the braid group. □

We denote by π the group homomorphism $U(H,c) \to H$, $[g] \mapsto g$. This notation is justified by the observation that $\pi(\Pi_{H,c}(\underline{g})) = \pi\underline{g}$ for all $\underline{g} \in c^n$. As in Proposition 2.4 (iii), one easily shows:

Proposition 4.3. *$U_1(H,c) := \ker \pi$ is a central subgroup of $U(H,c)$.*

Similarly, we define a group homomorphism $\mu_{H,c} \colon U(H,c) \to \mathbb{Z}^{D^*}$ via the following formula:

$$\forall g \in c, \quad \forall \gamma \in D^*, \quad \mu_{H,c}([g])(\gamma) = \begin{cases} 1 & \text{if } g \in \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

We have $\mu_{H,c}(\Pi_{H,c}(\underline{g})) = \mu_{H,c}(\underline{g})$ for all $\underline{g} \in c^n$: the (H,c)-multidiscriminant of a tuple can be retrieved from its (H,c)-lifting invariant, so the (H,c)-lifting invariant is a finer invariant.

Propositions 4.2 and 4.3 imply that $\Pi_{H,c}$ induces a monoid homomorphism from $\mathrm{Comp}(H,c)$ to the abelian group $U_1(H,c)$. We show an additional invariance property for the lifting invariant:

Proposition 4.4. *If $\gamma \in H$ and $x \in \mathrm{Comp}(H,c)$ then $\Pi_{H,c}(x) = \Pi_{H,c}(x^\gamma)$.*

Proof. Since c generates H, we may assume that $\gamma \in c$. In $U(H,c)$, we then have $[\gamma]\Pi_{H,c}(x) = \Pi_{H,c}(x^\gamma)[\gamma]$. By Proposition 4.3, the element $\Pi_{H,c}(x) \in U_1(H,c)$ is central, so we can cancel $[\gamma]$ in the equality to obtain $\Pi_{H,c}(x) = \Pi_{H,c}(x^\gamma)$. □

4.1.2. The structure of the group $U(H,c)$. We briefly summarize the results of [Woo21, Paragraph 2.1]. For this, define a group homomorphism $\widetilde{\pi} \colon \mathbb{Z}^{D^*} \to H^{\mathrm{ab}}$ as follows: when $\gamma \in D^*$ is a conjugacy class, let $\widetilde{\gamma}$ be the (well-defined) image in H^{ab} of the elements of $\gamma \in D^*$; then, for any $\psi \in \mathbb{Z}^{D^*}$, let

$$\widetilde{\pi}(\psi) := \prod_{\gamma \in D^*} \widetilde{\gamma}^{\psi(\gamma)}.$$

[Woo21, Theorem 2.5] states that the group $U(H,c)$ is isomorphic to the following fiber product:

$$U(H,c) \simeq S_c \underset{H^{\mathrm{ab}}}{\times} \mathbb{Z}^{D^*} \tag{4.1}$$

where the map $\mathbb{Z}^{D^*} \to H^{\mathrm{ab}}$ is $\widetilde{\pi}$, and S_c (a *reduced Schur cover* of H) is a finite group which fits in an exact sequence

$$1 \to H_2(H,c) \to S_c \to H \to 1 \tag{4.2}$$

for a specific quotient $H_2(H,c)$ of the second homology group $H_2(H,\mathbb{Z})$ of H. A consequence is that the central subgroup $U_1(H,c) := \ker(U(H,c) \to H)$ is isomorphic to a direct product:

$$U_1(H,c) \simeq H_2(H,c) \times \ker \widetilde{\pi}.$$

4.1.3. The lifting invariant distinguishes "big" components. The following result, proved in [EVW12, Theorem 7.6.1] and [Woo21, Theorem 3.1], implies that components are entirely determined by their (H, c)-lifting invariant as soon as each conjugacy class $\gamma \in D^*$ appears enough times in their (H, c)-multidiscriminant. This is a stronger version of a theorem of Conway–Parker–Fried–Völklein. To state the result, we introduce some notation. For any $\psi \in \mathbb{Z}^{D^*}$, we define $|\psi| \coloneqq \sum_{\gamma \in D^*} \psi(\gamma)$, and we let $\min \psi$ be the minimal value taken by $\psi(\gamma)$ for $\gamma \in D^*$. Then:

Theorem 4.5. *There is a constant $M_{H,c} \in \mathbb{N}$ such that, for any $\psi \in \mathbb{Z}^{D^*}$ satisfying $\min \psi \geq M_{H,c}$, the map $\Pi_{H,c}$ induces a bijection*

$$\left\{ \underline{g} \in G^{|\psi|} \;\middle|\; \begin{array}{c} g_1, \ldots, g_{|\psi|} \in c \\ \left\langle g_1, \ldots, g_{|\psi|} \right\rangle = H \\ \mu_{H,c}(\underline{g}) = \psi \end{array} \right\} \Big/ \mathrm{B}_{|\psi|} \xrightarrow{\sim} \left\{ x \in U(H, c) \;\middle|\; x \text{ has image } \psi \text{ in } \mathbb{Z}^{D^*} \right\}.$$

Restricting to product-one tuples yields, for each $\psi \in \ker \widetilde{\pi}$ satisfying $\min \psi \geq M_{H,c}$, a bijection

$$\left\{ x \in \mathrm{Comp}(H, c) \;\middle|\; \begin{array}{c} \langle x \rangle = H \\ \mu_{H,c}(x) = \psi \end{array} \right\} \xrightarrow{\sim} H_2(H, c).$$

4.2. The lifting invariant and fields of definition of glued components

In this subsection, we prove Theorem 1.2 (ii), using Theorem 4.5 and Theorem 3.1. First note there is a constant M independent of (H, c) satisfying the conclusion of Theorem 4.5:

$$M \coloneqq \max_{(H,c)} M_{H,c},$$

where the maximum is taken over couples (H, c) where H is a subgroup of G and c is a union of conjugacy classes of H which generates H. In what follows, the constant M is fixed in this way.

Definition 4.6. A tuple $\underline{g}$ of elements of G is *M-big* if every conjugacy class of $\left\langle \underline{g} \right\rangle$ appearing in $\underline{g}$ appears at least M times. A component $x \in \mathrm{Comp}(G)$ is *M-big* if its representing tuples are M-big.

For example, for any $x \in \mathrm{Comp}(G)$ and $k \geq M$, the component x^k is always M-big. Theorem 4.5 implies that an M-big component x is determined by its (H, c)-lifting invariant, where $H = \langle x \rangle$ and c is the union of the conjugacy classes of H which are monodromy classes of x. We now prove Theorem 4.7, whose third point is Theorem 1.2 (ii):

Theorem 4.7. *Let $x_1, \ldots, x_n \in \mathrm{Comp}(G)$ be components, and assume that $x_1 \cdots x_n$ is M-big. Then:*

(i) $\mathrm{ni}^\natural(x_1, \ldots, x_n) = \{x_1 \cdots x_n\}$.

(ii) For all $\sigma \in \Gamma_K$, we have $\sigma.(x_1 \cdots x_n) = (\sigma.x_1) \cdots (\sigma.x_n)$.

(iii) If $x_1, \ldots, x_n$ are defined over K, then $x_1 \cdots x_n$ is defined over K.

Proof. Let $H = \langle x_1 \cdots x_n \rangle$, and let c be the union of the conjugacy classes of H appearing in the component $x_1 \cdots x_n$. It follows from Proposition 4.4 and from the multiplicativity of $\Pi_{H,c}$ that all elements of $\mathrm{ni}^\natural(x_1, \ldots, x_n)$ have the same (H, c)-lifting invariant. Moreover, they are all M-big and have monodromy group H. By Theorem 4.5, these components are then all equal. This proves point (i). Points (ii) and (iii) follow from point (i) using Theorem 3.1, as in the proof of Theorem 3.3. □

Corollary 4.8. *Let $x_1, \ldots, x_n \in \mathrm{Comp}(G)$ be components defined over K. Then, for any $k \geq M$, the component $(x_1 \cdots x_n)^k$ is defined over K.*

4.3. The Galois action on lifting invariants

Let H be a subgroup of G, let c be a K-rational union of conjugacy classes of H generating H, and let D^* be the set of all conjugacy classes of H contained in c. In this subsection, we recall Wood's definition of a Γ_K-action on the set $U(H,c)$, taken from [Woo21, Paragraph 4.1]. That action describes how the Γ_K-action on components affects their lifting invariant (Proposition 4.9), thus generalizing the branch cycle lemma. Moreover, in Theorem 4.10, we show that the Galois action on lifting invariants is compatible with concatenation.

Consider a Galois automorphism $\sigma \in \Gamma_K$. Since c is K-rational, the $\chi(\sigma)$-th power operation defines a map $p_\sigma : D^* \to D^*$. For every conjugacy class $\gamma \in D^*$, fix an arbitrary element $g_\gamma \in \gamma$ and denote by $\widehat{g_\gamma}$ (resp. by $\widehat{(g_\gamma)^{\chi(\sigma^{-1})}}$) the element of S_c obtained by projecting $[g_\gamma] \in U(H,c)$ (resp. $\left[(g_\gamma)^{\chi(\sigma^{-1})}\right] \in U(H,c)$) via the isomorphism of Equation (4.1). The element

$$w(\gamma,\sigma) := \widehat{g_\gamma}^{-\chi(\sigma^{-1})}\widehat{(g_\gamma)^{\chi(\sigma^{-1})}}$$

is easily checked to be independent from the choice of g_γ, and belongs to the central subgroup $H_2(H,c) \subseteq S_c$ since it has trivial image in H.

Consider an element $v \in U(H,c)$, decomposed as (h,ψ) via the isomorphism $U(H,c) \simeq S_c \underset{H^{\mathrm{ab}}}{\times} \mathbb{Z}^{D^*}$ of Equation (4.1). In terms of those coordinates, we define:

$$\sigma.v := \left(h^{\chi(\sigma^{-1})} \prod_{\gamma \in D^*} w(\gamma,\sigma)^{\psi(\gamma)} \quad , \quad \psi \circ p_\sigma \right).$$

This defines a Γ_K-action on the set $U(H,c)$. The following fact follows from [Woo21, Paragraph 6.1]:

Proposition 4.9. *For any $x \in \mathrm{Comp}(H,c)$, we have $\Pi_{H,c}(\sigma.x) = \sigma.\Pi_{H,c}(x)$.*

By projection on $\mathbb{Z}^{D^*}$, Proposition 4.9 gives back the branch cycle lemma (Corollary 2.13 (i)). A consequence of Proposition 4.9 is the following refinement of Corollary 2.13 (ii): if a component $x \in \mathrm{Comp}(H,c)$ is defined over K, then its (H,c)-lifting invariant is Γ_K-invariant. Finally, we show that a product of Γ_K-invariant elements of $U_1(H,c)$ is Γ_K-invariant, implying that the lifting invariant cannot detect negative answers to Question 1.1. This boils down to the following fact:

Theorem 4.10. *The action of Γ_K on $U_1(H,c)$ is compatible with multiplication.*

Proof. Let $v, v' \in U_1(H,c)$, and decompose them as $v = (h,\psi)$, $v' = (h',\psi')$ with $h, h' \in H_2(H,c)$ and $\psi, \psi' \in \ker \widetilde{\pi}$. We have $vv' = (hh', \psi+\psi')$. Let $\sigma \in \Gamma_K$. With notation as above, we have

$$\begin{aligned}
\sigma.(vv') &= \left((hh')^{\chi(\sigma^{-1})} \prod_{\gamma \in D^*} w(\gamma,\sigma)^{(\psi+\psi')(\gamma)} \quad , \quad (\psi+\psi') \circ p_\sigma \right) \\
&= \left(h^{\chi(\sigma^{-1})}(h')^{\chi(\sigma^{-1})} \prod_{\gamma \in D^*} w(\gamma,\sigma)^{\psi(\gamma)} w(\gamma,\sigma)^{\psi'(\gamma)} \quad , \quad \psi \circ p_\sigma + \psi' \circ p_\sigma \right) \\
&= \left(\left(h^{\chi(\sigma^{-1})} \prod_{\gamma \in D^*} w(\gamma,\sigma)^{\psi(\gamma)} \right)\left((h')^{\chi(\sigma^{-1})} \prod_{\gamma \in D^*} w(\gamma,\sigma)^{\psi'(\gamma)} \right) \quad , \quad \psi \circ p_\sigma + \psi' \circ p_\sigma \right) \\
&= (\sigma.v)(\sigma.v').
\end{aligned}$$

□

Note that, in the computation, we have used the fact that $H_2(H,c)$ is abelian: the proof does not apply to arbitrary (non-commuting) elements of $U(H,c)$. Theorem 4.10 also implies positive answers to Question 1.1 in situations where the lifting invariant does determine components. For example, Theorem 4.7 (iii) could be deduced from Theorem 4.5 using Theorem 4.10 instead of Theorem 3.1.

5. The patching approach

In this section, we use Harbater's patching theory to prove Theorem 5.4, which is Theorem 1.2 (iii): if x and y are components defined over K, then $\mathrm{ni}^\natural(x, y)$ contains a component defined over K.

We first give a sketch of the argument, which also serves as an outline of the section:

In Subsection 5.1, we construct infinitely many extensions K_1, K_2, ... of K, pairwise linearly disjoint, over which the components x and y both have points (Lemma 5.2). This is accomplished by using Hilbert's irreducibility theorem (Theorem 5.1) repeatedly. For each $n \in \mathbb{N}$, denote by f_n (resp. g_n) a $K_n((t))$-point of x (resp. y) obtained from a K_n-point of x (resp. y). Note that $K_n((t))$ is a complete valued field for the (t)-adic valuation.

In Subsection 5.2, we prove that for each $n \in \mathbb{N}$, the cover obtained by patching the $K_n((t))$-G-covers f_n and g_n is a $K_n((t))$-G-cover lying in a component $m_n \in \mathrm{ni}^\natural(x, y)$ (Lemma 5.3). In particular, the field of definition of the component m_n is contained in $\overline{\mathbb{Q}} \cap K_n((t)) = K_n$.

Finally, we observe that two components m_n, $m_{n'}$ must be equal as $\mathrm{ni}^\natural(x, y)$ is finite. The component $m_n = m_{n'}$ is then defined over $K_n \cap K_{n'} = K$: this is precisely Theorem 5.4. The detailed proof is the focus of Subsection 5.3.

The results of this section rely crucially on the fact that number fields are Hilbertian.

5.1. Constructing covers with linearly disjoint fields of definition

In this subsection, we prove Lemma 5.2, which is used in the proof of Theorem 5.4. This lemma lets us construct points in a given component whose fields of definitions are linearly disjoint over the field of definition of the component. The proof uses the following form of Hilbert's irreducibility theorem:

Theorem 5.1 (Hilbert's irreducibility theorem)**.** *Let $L'|L$ be a finite extension of number fields and $p\colon X \to Y$ be a finite étale morphism from a variety X over L to an open subvariety Y of $\mathbb{A}^n_L$. Assume that $X_{L'}$ is irreducible. Then there exists an L-point $t \in Y(L)$ such that the $\overline{\mathbb{Q}}$-points of X in the fiber above t lie in a single $\mathrm{Gal}(\overline{\mathbb{Q}}|L')$-orbit.*[5]

When $L = L'$, this theorem is well-known (see [Ser92, Theorem 3.4.1 and Proposition 3.3.1]). The fact that L' may be chosen larger than L follows from [FJ23, Corollary 12.2.3]. We now state and prove the lemma:

Lemma 5.2. *Let S be a geometrically irreducible component of the Hurwitz scheme $\mathcal{H}^*(G, n)_{\overline{\mathbb{Q}}}$, and let L be a number field. Then, the following properties are equivalent:*

(i) S is defined over L.

(ii) For every finite extension $L'|L$, there is a finite Galois extension $\widetilde{L}|L$ such that $\widetilde{L}$ and L' are linearly disjoint over L, and such that S has an $\widetilde{L}$-rational point.

(iii) There exist infinitely many finite Galois extensions of L, pairwise linearly disjoint, over which S has points;

(iv) There exist two extensions L_1, L_2 of L over which S has points and such that $L_1 \cap L_2 \cap \overline{\mathbb{Q}} = L$.

Proof.

(i) ⇒ (ii) S being defined over L, we see it as an irreducible component of $\mathcal{H}^*(G, n)_L$. Replacing L' by its Galois closure over L if needed, we may assume that $L'|L$ is Galois. Since S is geometrically irreducible, its extension of scalars $S_{L'}$ is irreducible. The branch point morphism $S \to (\mathrm{Conf}_n)_L$

[5] In terms of scheme-theoretic points, this means the following: there is a closed point τ of Y with residue field L such that the scheme-theoretic fiber $p^{-1}(\tau)$ consists of a single (closed) point, and remains a single point even after extending scalars to L'.

is finite étale. By Hilbert's irreducibility theorem (Theorem 5.1), there is a configuration $\underline{t} \in \mathrm{Conf}_n(L)$ such that the fiber $F \subset S(\overline{\mathbb{Q}})$ above $\underline{t}$ consists of a single $\mathrm{Gal}(\overline{\mathbb{Q}}|L')$-orbit.

Since the fiber F consists of a single $\mathrm{Gal}(\overline{\mathbb{Q}}|L')$-orbit (and thus also of a single $\mathrm{Gal}(\overline{\mathbb{Q}}|L)$-orbit), it corresponds to a single closed (scheme-theoretic) point x of S which, after extending scalars to L', remains a single closed point of $S_{L'}$. The fact that the closed point x of S remains a single point after extending scalars to L' means that its residue field $\widetilde{L}$ (which is the smallest extension of L over which the points of F are rational) remains a field after taking the tensor product with L' over L, i.e., that $\widetilde{L}$ and L' are linearly disjoint over L.

(ii) ⇒ (iii) Choose a geometric point of S and denote by L_1 a finite Galois extension of L over which that point is rational. Use (ii) with $L' = L_1$ to obtain a new extension $L_2|L$, linearly disjoint with L_1, over which S has a point. Apply (ii) again with $L' = L_1L_2$, etc. We obtain countably many pairwise linearly disjoint finite Galois extensions $L_1, L_2, \dots$ of L over which S has points.

(iii) ⇒ (iv) Clear.

(iv) ⇒ (i) The field of definition L' of S is a number field contained in the field of definition of any point. Hence, $L' \subseteq L_1 \cap L_2 \cap \overline{\mathbb{Q}} = L$, i.e., S is defined over L. □

5.2. Relating patching and gluing

In this section, we prove Lemma 5.3, which is used in the proof of Theorem 5.4. The lemma contains all the results from Harbater's patching theory which we need for the proof.

Let $\mathcal{O}$ be a complete discrete valuation ring of characteristic zero, and let $L := \mathrm{Frac}(\mathcal{O})$ be its field of fractions, which is a complete non-Archimedean valued field. Since $\overline{\mathbb{Q}}$ is algebraically closed and $\overline{L}$ contains $\overline{\mathbb{Q}}$, extension of scalars induces a bijection between the connected components of $\mathcal{H}^*(G,n)_{\overline{\mathbb{Q}}}$ and those of $\mathcal{H}^*(G,n)_{\overline{L}}$ (see [Stacks, Lemma 0363]). This bijection allows us to implicitly identify the components of $\mathcal{H}^*(G,n)_{\overline{\mathbb{Q}}}$ with those of $\mathcal{H}^*(G,n)_{\overline{L}}$. For instance, we say that a component $x \in \mathrm{Comp}(G)$ has an L-rational point if its extension of scalars to $\overline{L}$ has an L-rational point.

Lemma 5.3. *Let $x_1, x_2 \in \mathrm{Comp}(G)$ be components which have L-rational points. Then, there is a component $y \in \mathrm{ni}^\natural(x_1, x_2)$ which has an L-rational point.*

Proof. The proof is in four steps:

- **Step 1: Setting things up**

  For each $i \in \{1,2\}$, let $r_i = \deg(x_i)$, $G_i = \langle x_i \rangle$ and fix an L-model $f_i \in \mathcal{H}^*(G,r_i)(L)$ of an L-rational point of x_i, corresponding to an L-G-cover with a marked L-point above ∞ (cf. Paragraph 2.3.2). In the cover f_i, keep only the geometrically connected component of the marked point, which is defined over L since the marked point is L-rational. This turns f_i into a geometrically connected L-G_i-cover with a marked L-point. The cover f_i belongs to the component x'_i of $\mathcal{H}^*(G_i,r_i)_L$ obtained by keeping only the component of the marked points in the covers of x_i, like in Paragraph 2.3.4. Without loss of generality, we may assume that $G = \langle G_1, G_2 \rangle$.

- **Step 2: Patching covers over L**

  We use the algebraic patching results of [HV96]. First define:

$$L\{z\} := \left\{ \textstyle\sum_{i\geq 0} a_i z^i \in L[\![z]\!] \;\middle|\; a_i \underset{i\to\infty}{\to} 0 \right\} \qquad Q_1 := \mathrm{Frac}(L\{z\})$$
$$L\{z^{-1}\} := \left\{ \textstyle\sum_{i\geq 0} a_i z^{-i} \in L[\![z^{-1}]\!] \;\middle|\; a_i \underset{i\to\infty}{\to} 0 \right\} \qquad Q_2 := \mathrm{Frac}(L\{z^{-1}\})$$
$$L\{z,z^{-1}\} := \left\{ \textstyle\sum_{i\in\mathbb{Z}} a_i z^i \in L[\![z,z^{-1}]\!] \;\middle|\; a_i \underset{i\to\pm\infty}{\to} 0 \right\} \qquad \widehat{Q} := \mathrm{Frac}(L\{z,z^{-1}\}).$$

Let also $Q'_1 = Q_2$ and $Q'_2 = Q_1$. From the point of view of rigid analytic geometry, Q_1 (resp. Q_2, and $\widehat{Q}$) is the algebra of analytic functions on the unit disk D_1 centered at 0 (resp. a disk D_2 centered at ∞, and the annulus $D_1 \cap D_2$):

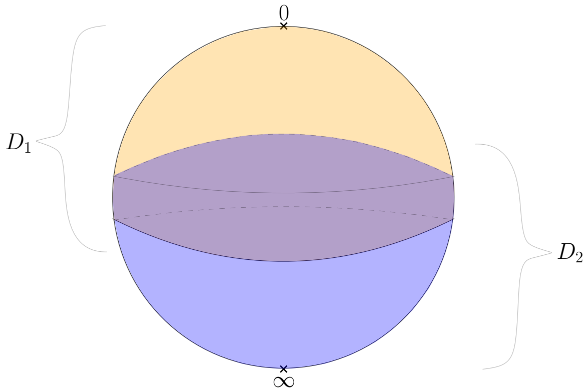
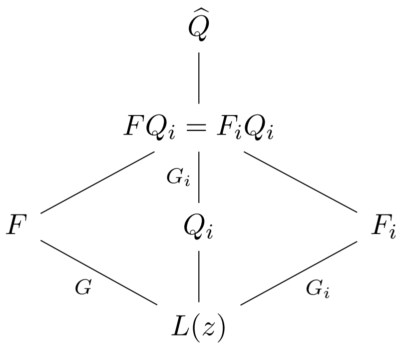


Figure 1: The rigid analytic projective line

The marked points of the G-covers f_1 and f_2 are L-points in an unramified fiber. Their existence ensures that the corresponding field extensions F_1 and F_2 of $L(z)$ have an unramified prime of degree 1. By [HV96, Lemma 4.2], for each $i \in \{1, 2\}$, we can then replace f_i by an isomorphic L-G_i-cover such that F_i is contained in Q'_i, and in particular the branch locus $\underline{t}_i \in \mathrm{Conf}_{r_i}(L)$ of f_i is included in a disk strictly smaller than D_i. Then, [HV96, Proposition 4.3] implies that f_1 and f_2 can be patched into a geometrically connected L-G-cover f with an L-point.

- **Step 3: Restriction of the patched cover f to disks**

  Denote by F the field extension corresponding to f, i.e., the *compound* of F_1 and F_2 in the terminology of [HV96], which is a subfield of $\widehat{Q}$. By [HV96, Lemma 3.6 (b)], we have the equalities $FQ_i = F_iQ_i$ (for each $i \in \{1, 2\}$) inside $\widehat{Q}$. Moreover, the group homomorphism $\mathrm{Gal}(FQ_i|Q_i) \to \mathrm{Gal}(F|L(z))$ corresponds to the inclusion $G_i \hookrightarrow G$. We sum this up by the following diagram:



  Geometrically, the equality $FQ_1 = F_1Q_1$ means that the cover f_1 is isomorphic to f as a rigid analytic cover when both are restricted to the unit disk D_1, and similarly for f_2 and f in restriction to D_2. In consequence, the branch points of f are given by the configuration $\underline{t} = \underline{t}_1 \cup \underline{t}_2$. Let y be the component of $\mathcal{H}^*(G, r_1+r_2)_{\overline{\mathbb{Q}}}$ whose extension of scalars to $\overline{L}$ contains f as an $\overline{L}$-point. To show that the component y fits, it remains to check that $y \in \mathrm{ni}(x_1, x_2)$.

- **Step 4: Admissibility of the special fiber $\overline{f}$ of the patched cover**

  For each $i \in \{1, 2\}$, since $\underline{t}_i$ is included in a disk strictly smaller than D_i, the configuration $\underline{t}_i$ is mapped to a single element $\overline{a_i}$ modulo the maximal ideal of $\mathcal{O}$, with $\overline{a_1} \neq \overline{a_2}$. The projective line $\mathbb{P}^1_L$, marked by $\underline{t} = \underline{t}_1 \cup \underline{t}_2$, has a semistable model $\widetilde{P}_{\underline{t}}$ over $\mathcal{O}$, whose special fiber $\overline{P}_{\underline{t}}$ is a "comb" with two teeth T_1, T_2, one for each coset $\overline{a_1}, \overline{a_2}$. For $i \in \{1,2\}$, the points of the configuration $\underline{t}_i$ extend to sections which specialize to r_i distinct nonsingular points of the tooth T_i.

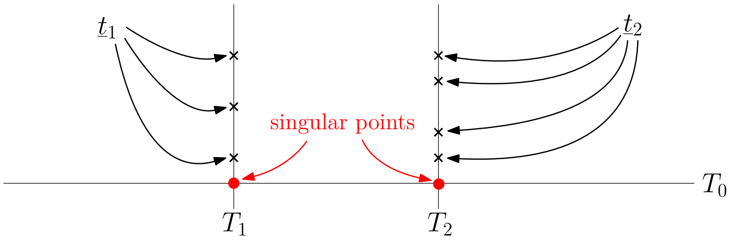


Figure 2: The comb with two teeth $\overline{P}_{\underline{t}}$

The cover f, branched at $\underline{t}$, extends to a cover $\widetilde{f}$ of the semistable model $\widetilde{P}_{\underline{t}}$, which is ramified along the sections of the points in $\underline{t}$. The special fiber $\overline{f}$ of $\widetilde{f}$ is a cover of the comb which lies on the "boundary" of the component y in the sense of the Wewers' compactification, see [DE06, Paragraph 1.2] or [Cau12, Paragraph 3.3.1].

To prove that the special fiber $\overline{f}$ of $\widetilde{f}$ is unramified at the singular points of the comb, we follow [DE06, Paragraph 2.3] closely. The restriction of f to D_1 extends to a cover of the rigid projective line which has no branch points outside D_1 (namely, f_1). By the arguments of [DE06, Proposition 2.3, (ii)⇒(i)⇒(iii)], the restricted cover $f_{|D_1}$ is trivial above the annulus ∂D_1. The same holds for $f_{|D_2}$ above the annulus ∂D_2. Hence, $\overline{f}$ is unramified at the singular points of the comb.

We conclude that $\overline{f}$ is a cover of the comb $\overline{P}_{\underline{t}}$ unramified at the singular points, whose restriction to the i-th tooth is isomorphic to the cover f_i (which belongs to the component x'_i). This implies that $\overline{f}$ is a Δ-admissible cover in the sense of [Cau12, Definition 3.7], where

$$\Delta = \Big(G, (G_1, G_2), (x'_1, x'_2)\Big)$$

is the degenerescence structure associated to (x'_1, x'_2). By [Cau12, Proposition 3.9], the component y containing f is a Δ-component, which in our terminology means that $y \in \mathrm{ni}(x_1, x_2)$ as we noted in Subsection 3.1. □

5.3. Proof of the theorem

We finally prove Theorem 5.4, which is Theorem 1.2 (iii). For this, we use Lemmas 5.2 and 5.3, and we follow the outline of the proof given at the beginning of this section.

Theorem 5.4. *Let $x, y \in \mathrm{Comp}(G)$ be components defined over K. Then $\mathrm{ni}^{\natural}(x,y)$ contains a component defined over K.*

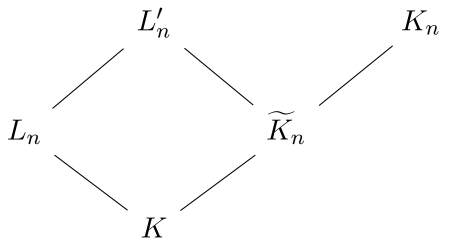
Proof. Let $r_1 = \deg(x)$ and $r_2 = \deg(y)$. Since the components x and y are defined over K, we fix geometrically irreducible K-models $X \subseteq \mathcal{H}^*(G, r_1)_K$ and $Y \subseteq \mathcal{H}^*(G, r_2)_K$ of x and y.

We inductively construct two sequences of marked G-covers $(f_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ and an infinite sequence of field extensions $K_n|K$ such that:

- K_n is linearly disjoint with the Galois closure of the compositum $K_1 \cdots K_{n-1}$ over K.
- f_n and g_n are K_n-points of X and Y respectively.

For f_1 and g_1, choose arbitrary $\overline{\mathbb{Q}}$-points of X and Y respectively, and let K_1 be the smallest extension of K over which they are both rational.

Assume that we have constructed $K_1, \dots, K_{n-1}$ and $f_1, g_1, \dots, f_{n-1}, g_{n-1}$. Let L_n be the Galois closure of the compositum $K_1 \cdots K_{n-1}$ over K. Apply Lemma 5.2 [(i) $\Rightarrow$ (ii)] with $(L, L', S) = (K, L_n, X)$ to obtain a finite extension $\widetilde{K_n}|K$, linearly disjoint with L_n over K, and a $\widetilde{K_n}$-point f_n of X. Let L'_n be the Galois closure (over K) of the compositum $L_n\widetilde{K_n}$ and apply Lemma 5.2 [(i) $\Rightarrow$ (ii)] again, with $(L, L', S) = (\widetilde{K_n}, L'_n, Y_{\widetilde{K_n}})$ to obtain a finite extension $K_n|\widetilde{K_n}$, linearly disjoint with L'_n over $\widetilde{K_n}$, and a K_n-point g_n of Y. Finally, replace the $\widetilde{K_n}$-point f_n by the corresponding K_n-point. The inclusions between the fields defined above are summed up by the following diagram:



By construction, we have $f_n \in X(K_n)$ and $g_n \in Y(K_n)$. Now:

$$K_n \cap L_n = K_n \cap (L'_n \cap L_n) = (K_n \cap L'_n) \cap L_n = \widetilde{K_n} \cap L_n = K.$$

Since $L_n|K$ is Galois, this implies that K_n and L_n are linearly disjoint over K. We have shown that the constructed sequences (f_n), (g_n), (K_n) satisfy the desired properties.

Next, we show that for each n there is a component $z_n \in \mathrm{ni}^\natural(x, y)$ defined over K_n. Denote by $\widetilde{f_n}$ (resp. $\widetilde{g_n}$) the $K_n(\!(t)\!)$-point of X (resp. Y) obtained by seeing $f_n \in X(K_n)$ (resp. $g_n \in Y(K_n)$) as a $K_n(\!(t)\!)$-point. Since $F = K_n(\!(t)\!)$ is a complete valued non-Archimedean field (for the (t)-adic valuation), Lemma 5.3 implies that there is a component $z_n \in \mathrm{ni}^\natural(x, y)$ which has a $K_n(\!(t)\!)$-rational point. In particular, the field of definition of the component z_n is contained in $K_n(\!(t)\!) \cap \overline{\mathbb{Q}} = K_n$.

Finally, since $\mathrm{ni}^\natural(x, y)$ is finite, there are distinct integers $n \neq n'$ such that $z_n = z_{n'}$. The field of definition of the component z_n is then contained in $K_n \cap K_{n'} = K$: we have found a component $z_n \in \mathrm{ni}^\natural(x, y)$ defined over K. □

5.4. Applications of Theorem 5.4

In this subsection, we give applications of Theorem 5.4: this patching result allows one to construct components defined over $\mathbb{Q}$ with few branch points for many groups of interest.

Example 5.5. The Mathieu group M_{23} is the only sporadic simple group not known to be a Galois group over $\mathbb{Q}$. In [Cau16, Exemple 3.12], a component defined over $\mathbb{Q}$ of connected M_{23}-covers with 15 branch points is constructed using a patching method. Theorem 5.4 improves upon this result. The group M_{23} is generated by two conjugate elements a, a^h of order 3. Using GAP:

```
a := (1, 22, 14) (2, 13, 9) (3, 8, 6) (7, 16, 21) (10, 18, 19) (11, 23, 12);
h := (1,17)(3,21,7)(4,13)(5,22,15,23,9,16)(6,10,11,8,20,19)(12,14,18) ;
StructureDescription(Group(a, h^(-1) * a * h)); # Output: "M23"
```

By the conclusions of Example 2.14, the component $x := (a, a^{-1})$ and its conjugate x^h are defined over $\mathbb{Q}$. By Theorem 5.4, there are elements $\gamma, \gamma' \in M_{23}$ such that $x^\gamma x^{\gamma' h}$ is a component defined over $\mathbb{Q}$ of connected M_{23}-covers with four branch points. The same is true of the component $xx^{\widetilde{\gamma}}$ where $\widetilde{\gamma} = \gamma^{-1}\gamma' h$. However, we know little about $\widetilde{\gamma} \in M_{23}$.

It should be noted that there are other ways to construct such components. For example, in [Kön14, Section 5.3], it is mentioned that there are 980 isomorphism classes of unmarked connected

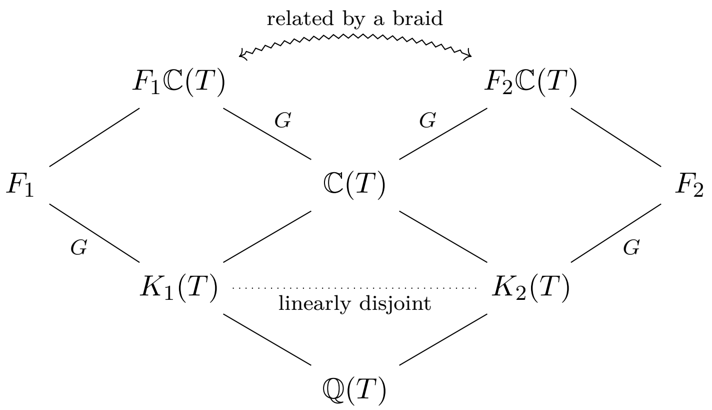
M_{23}-covers with four specific rational branch points having a specific rational multidiscriminant (the monodromy elements have orders 2, 2, 3 and 5). Moreover, these covers belong to the same connected component, i.e., the braid group B_4 acts transitively on this 980-element set. By rigidity, this component is defined over $\mathbb{Q}$, and König finds equations defining this component as well as rational points on its boundary.

Our results have a different flavor. Because of the infinite pigeonhole argument used to prove Theorem 5.4, it is unlikely to write equations for the component constructed. However, since they do not require checking a braid-transitivity condition, our results are very general. Indeed, Example 5.5 belongs to a large family: components defined over $\mathbb{Q}$ of connected G-covers with 4 branch points exist for any group G generated by two elements with orders in $\{2, 3, 4, 6\}$. More generally:

Proposition 5.6. *Let G be a group generated by elements $g_1, \ldots, g_n$. Denote by $m(i)$ the number of elements of order i in this generating set. Then there is a component defined over $\mathbb{Q}$ of connected G-covers whose number of branch points is $2m(2) + \sum_{i\geq 3} \varphi(i)m(i)$, where φ is Euler's totient function.*

Proof. For each $i \in \{1, \ldots, n\}$, define a tuple $\underline{g}_i$ in the following way: if $\mathrm{ord}(g_i) = 2$, then $\underline{g}_i = (g_i, g_i)$; otherwise $\underline{g}_i = (g_i^{k_1}, g_i^{k_2}, \ldots, g_i^{k_{\varphi(\mathrm{ord}(g_i))}})$, where $k_1 < k_2 < \cdots < k_{\varphi(\mathrm{ord}(g_i))}$ are the integers of $\{1, \ldots, \mathrm{ord}(g_i) - 1\}$ coprime to $\mathrm{ord}(g_i)$. We have $\pi \underline{g}_i = 1$, so the tuple $\underline{g}_i$ defines a component whose group is the abelian group $\langle g_i \rangle$. This component is defined over $\mathbb{Q}$ by Corollary 2.13 (iii). By Theorem 5.4, some component of the form $\prod_{i=1}^{n} (\underline{g}_i)^{\gamma_i}$ is defined over $\mathbb{Q}$ and has monodromy group $\langle g_1, \ldots, g_n \rangle = G$. The size of such a component is $\sum_{i=1}^{n} 2m(2) + \sum_{i\geq 3} \varphi(i)m(i)$. □

Note that Lemma 5.2 implies the following: every component of group G defined over $\mathbb{Q}$ contains two connected G-covers which are defined over two linearly disjoint Galois number fields K_1 and K_2:



6. Reduction of the Galois action to components of small size

In this section, we give a final application of the ideas of Sections 3 and 4: we express the Galois action on all components in terms of the action on components of small size (Proposition 6.1). Let $\psi(G)$ be the sum of the orders of the elements of G:

$$\psi(G) \coloneqq \sum_{g \in G} \mathrm{ord}(g).$$

Consider an n-tuple $\underline{g} = (g_1, \ldots, g_n) \in G^n$, and let $H = \left\langle \underline{g} \right\rangle$. If $n > \psi(G)$, then there is an element $g \in G$ which appears at least $\mathrm{ord}(g) + 1$ times in the tuple $\underline{g}$. Usual braid manipulations allow one to move $\mathrm{ord}(g)$ of these occurrences of g to the beginning of the tuple, i.e., we have the following equality in $\mathrm{Comp}(G)$:

$$\underline{g} = (\underbrace{g, \ldots, g}_{\mathrm{ord}(g)})y$$

for some $y \in \mathrm{Comp}(G)$, which has monodromy group H (we have made sure that at least one occurrence of g was untouched). Note that $(g, \dots, g)$ and y are permuting components and that $\langle (g, \dots, g) \rangle = \langle g \rangle$ is abelian. We have:

$$\begin{aligned} \sigma.\underline{g} &= (\sigma.(g, \dots, g))\,(\sigma.y) && \text{by Theorem 3.3 (ii)} \\ &= \left(g^{\chi(\sigma^{-1})}, \dots, g^{\chi(\sigma^{-1})}\right)(\sigma.y) && \text{by Corollary 2.13 (i).} \end{aligned}$$

We can iterate this factorization process until the size of y is at most $\psi(G)$. This shows that the Galois action on components is entirely determined by the cyclotomic character and by the Galois action on components of size $\leq \psi(G)$. We turn this into a precise proposition:

Proposition 6.1. *Let $x \in \mathrm{Comp}(G)$ be a component and $H = \langle x \rangle$. There are elements $g_1, \dots, g_r \in H$ and a component $y \in \mathrm{Comp}(G)$ of group H with $\deg(y) \leq \psi(G)$ such that:*

$$x = \left(\prod_{i=1}^{r} (\underbrace{g_i, \dots, g_i}_{\mathrm{ord}(g_i)}) \right) y.$$

Moreover, once x is expressed under this form, its image under the action of any $\sigma \in \Gamma_K$ is described in terms of the cyclotomic character χ and of the Galois action on components of size $\leq \psi(G)$:

$$\sigma.x = \left(\prod_{i=1}^{r} \Big(\underbrace{g_i^{\chi(\sigma^{-1})}, \dots, g_i^{\chi(\sigma^{-1})}}_{\mathrm{ord}(g_i)}\Big) \right) (\sigma.y).$$

We give another example of this phenomenon. Let H be a subgroup of G and c a K-rational union of conjugacy classes of H. Denote by $\mathcal{C}_{H,c}$ the set of components $x \in \mathrm{Comp}(H, c)$ such that $\langle x \rangle = H$, and whose (H, c)-lifting invariant is Γ_K-invariant. Then:

Proposition 6.2. *Assume that every component $x \in \mathcal{C}_{H,c}$ of size $\leq 2\,|c|\,\psi(G)$ is defined over K. Then, every component $x \in \mathcal{C}_{H,c}$ is defined over K.*

Proof. We reason by induction. Consider a component $x \in \mathcal{C}_{H,c}$ of size $n > 2\,|c|\,\psi(G)$, and assume that every component in $\mathcal{C}_{H,c}$ of size $< n$ is defined over K. Choose a tuple $\underline{g} \in c^n$ representing x. Since $n > 2\,|c|\,\psi(G)$, there is some $g \in c$ which appears at least $2\mathrm{ord}(g)\,|c| + 1$ times in $\underline{g}$. As x has a Γ_K-invariant (H, c)-lifting invariant, it also has a Γ_K-invariant (H, c)-multidiscriminant. (This follows directly from the definition of the Γ_K-action on lifting invariants in Subsection 4.3.)

Let $g_1, \dots, g_r$ be the distinct elements of G obtained as $g^{\chi(\sigma)}$ for some $\sigma \in \Gamma_K$. By Corollary 2.13 (iii), the following component, whose group is the abelian group $\langle g \rangle$, is defined over K:

$$y \coloneqq (\underbrace{g_1, \dots, g_1}_{\mathrm{ord}(g)}, \underbrace{g_2, \dots, g_2}_{\mathrm{ord}(g)}, \dots, \underbrace{g_r, \dots, g_r}_{\mathrm{ord}(g)}).$$

To prove that there is a component z with $\langle z \rangle = H$ such that $x = yz$, we apply the factorization lemma [Seg24, Lemma 3.6]. Consider a conjugacy class γ of H appearing in y. Then:

- The conjugacy class of g appears at least $2\mathrm{ord}(g)\,|c| + 1$ times in $\underline{g}$, because g itself does. The conjugacy class γ is some $\chi(\sigma)$-th power of that conjugacy class, and x has a K-rational (H, c)-multidiscriminant, so $\mu_{H,c}(x)(\gamma) \geq 2\mathrm{ord}(g)\,|c| + 1$.
- The conjugacy class γ appears at most $\mathrm{ord}(g)\,|c|$ times in y since $\deg(y) \leq \mathrm{ord}(g)\,|c|$.

Hence:

$$\begin{aligned}\mu_{H,c}(x)(\gamma) &\geq 2\mathrm{ord}(g)\,|c| + 1 \\ &\geq \mathrm{ord}(g)(|\gamma| + |c|) \\ &= \mathrm{ord}(\gamma)\,|\gamma| + \mathrm{ord}(g)\,|c| \\ &\geq \mathrm{ord}(\gamma)\,|\gamma| + \mu_{H,c}(y)(\gamma).\end{aligned}$$

By [Seg24, Lemma 3.6], there exists $z \in \mathrm{Comp}(G)$ such that $x = yz$ and $\langle z \rangle = H$. The equality $x = yz$ implies that $z \in \mathrm{Comp}(H, c)$ and that

$$\Pi_{H,c}(x) = \Pi_{H,c}(y)\Pi_{H,c}(z). \tag{6.1}$$

For any $\sigma \in \Gamma_K$, Theorem 4.10 then implies:

$$\sigma.\Pi_{H,c}(x) = (\sigma.\Pi_{H,c}(y))(\sigma.\Pi_{H,c}(z)),$$

i.e., as $x, y \in \mathcal{C}_{H,c}$:

$$\Pi_{H,c}(x) = \Pi_{H,c}(y)(\sigma.\Pi_{H,c}(z)). \tag{6.2}$$

Since the lifting invariant takes values in a group, Equation (6.1) and Equation (6.2) together imply that $\Pi_{H,c}(z) = \sigma.\Pi_{H,c}(z)$. Hence, $z \in \mathcal{C}_{H,c}$. By the induction hypothesis, z is defined over K. Moreover, $\langle y \rangle \subseteq H$ so y and z are permuting, and thus $x = yz$ is defined over K by Theorem 3.3 (iii). We conclude by induction. □

References

[Cau12] Orlando Cau. "Delta-composantes des espaces de modules de revêtements". In: *Journal de Théorie des Nombres de Bordeaux* 24.3 (2012), pp. 557–582. DOI: 10.5802/jtnb.811.

[Cau16] Orlando Cau. "Delta-composantes des modules de revêtements : corps de définition". In: *Bulletin de la Société Mathématique de France* 144.2 (2016), pp. 145–162. URL: https://smf.emath.fr/system/files/2017-08/smf_bull_144_145-162.pdf.

[DD97] Pierre Dèbes and Jean-Claude Douai. "Algebraic covers: Field of moduli versus field of definition". In: *Annales Scientifiques de l'École Normale Supérieure. Quatrième Série* 30.3 (1997), pp. 303–338. DOI: 10.1016/S0012-9593(97)89922-3.

[DE06] Pierre Dèbes and Michel Emsalem. "Harbater-Mumford Components and Towers of Moduli Spaces". In: *Journal of the Institute of Mathematics of Jussieu* 5.3 (2006), pp. 351–371. DOI: 10.1017/S1474748006000053.

[Ems95] Michel Emsalem. "Familles de revêtements de la droite projective". In: *Bulletin de la Société Mathématique de France* 123.1 (1995), pp. 47–85. DOI: 10.24033/bsmf.2250.

[EVW12] Jordan S. Ellenberg, Akshay Venkatesh, and Craig Westerland. "Homological stability for Hurwitz spaces and the Cohen-Lenstra conjecture over function fields, II." **Withdrawn preprint**. 2012. arXiv: 1212.0923v1 [math.NT].

[EVW16] Jordan S. Ellenberg, Akshay Venkatesh, and Craig Westerland. "Homological stability for Hurwitz spaces and the Cohen-Lenstra conjecture over function fields". In: *Annals of Mathematics. 2^{nd} Series* 183.3 (2016), pp. 729–786. DOI: 10.4007/annals.2016.183.3.1.

[FJ23] Michael D. Fried and Moshe Jarden. *Field arithmetic*. Vol. 11. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. Springer, 2023. ISBN: 978-3-031-28019-1. DOI: 10.1007/978-3-031-28020-7.

[Fri77] Michael D. Fried. "Fields of definition of function fields and Hurwitz families. Groups as Galois groups". In: *Communications in Algebra* 5.1 (1977), pp. 17–82. DOI: 10.1080/00927877708822158.

[FV91] Michael D. Fried and Helmut Völklein. "The inverse Galois and rational points on moduli spaces". In: *Mathematische Annalen* 290.4 (1991), pp. 771–800. DOI: 10.1007/BF01459271.

[Har03] David Harbater. "Patching and Galois theory". In: *Galois groups and fundamental groups*. Ed. by Leila Schneps. Cambridge: Cambridge University Press, 2003, pp. 313–424. ISBN: 0-521-80831-6. URL: https://www2.math.upenn.edu/~harbater/patch35.pdf.

[Har83] Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics. Springer, 1983. ISBN: 978-0-387-90244-9.

[HV96] Dan Haran and Helmut Völklein. "Galois groups over complete valued fields". In: *Israel Journal of Mathematics* 93 (1996), pp. 9–27. DOI: 10.1007/BF02761092.

[Kan24a] Vassil Kanev. "Hurwitz moduli varieties parameterizing Galois covers of an algebraic curve". In: *Serdica Mathematical Journal* 50.1 (2024), pp. 47–102. DOI: 10.55630/serdica.2024.50.47-102.

[Kan24b] Vassil Kanev. *Hurwitz moduli varieties parameterizing pointed covers of an algebraic curve with a fixed monodromy group*. 2024. arXiv: 2403.12756 [math.AG].

[Kön14] Joachim König. "The inverse Galois problem and explicit computation of families of covers of $\mathbb{P}^1\mathbb{C}$ with prescribed ramification". PhD thesis. Institut für Mathematik der Universität Würzburg, Mar. 2014. URL: https://opus.bibliothek.uni-wuerzburg.de/files/10014/Koenig_Joachim_Dissertation.pdf.

[Liu95] Qing Liu. "Tout groupe fini est un groupe de Galois sur $\mathbb{Q}_p(T)$". In: *Recent developments in the Inverse Galois Problem*. Ed. by M. D. Fried. Vol. 186. Contemporary Mathematics. American Mathematical Society, 1995, pp. 261–265.

[RW06] Matthieu Romagny and Stefan Wewers. "Hurwitz Spaces". In: *Séminaire et Congrès* 13 (2006), pp. 313–341. URL: https://perso.univ-rennes1.fr/matthieu.romagny/articles/hurwitz_spaces.pdf.

[Seg23] Béranger Seguin. "Géométrie et arithmétique des composantes des espaces de Hurwitz". PhD thesis. Université de Lille, 2023. URL: http://beranger-seguin.fr/these.pdf.

[Seg24] Béranger Seguin. *Counting Components of Hurwitz Spaces*. 2024. arXiv: 2409.18246 [math.AT]. To appear in *Israel Journal of Mathematics*.

[Ser92] Jean-Pierre Serre. *Topics in Galois theory*. Vol. 1. Research Notes in Mathematics. 1992. ISBN: 0-86720-210-6.

[Stacks] The Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu.

[Sza09] Tamás Szamuely. *Galois groups and fundamental groups*. Vol. 117. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 2009. DOI: 10.1017/CBO9780511627064.

[Woo21] Melanie Wood. "An algebraic lifting invariant of Ellenberg, Venkatesh, and Westerland". In: *Research in the Mathematical Sciences* 8 (June 2021). DOI: 10.1007/s40687-021-00259-2. URL: https://math.berkeley.edu/~mmwood/Publications/lifting.pdf.