The Geometry of Rings of Components of Hurwitz Spaces

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Abstract. We consider a variant of the ring of components of Hurwitz spaces introduced by Ellenberg, Venkatesh and Westerland. By focusing on Hurwitz spaces classifying covers of the projective line, the resulting ring of components is commutative, which lets us study it from the point of view of algebraic geometry and relate its geometric properties to the numerical invariants involved in our previously obtained asymptotic counts. Specifically, we describe a stratification of the prime spectrum of the ring of components, and we compute the dimensions and degrees of the strata. Using the stratification, we give a complete description of the spectrum in some cases.

MSC 2020: 14A10 · 13A02 · 16S34

Contents

1. Introduction and main results

For the whole article, we fix a finite group G, a nonempty set D of nontrivial conjugacy classes of G, a map $\xi\colon D \to \mathbb{Z}_{>0}$ (attributing a multiplicity to each conjugacy class $\gamma \in D$), and a field k whose characteristic does not divide the order $|G|$ of the group G.

1.1. Context

In [EVW16], Ellenberg, Venkatesh and Westerland introduced the *ring of components of Hurwitz spaces*, a graded algebra whose elements are linear combinations of connected components of Hurwitz spaces parametrizing marked G-covers[1] of the affine line. The grading of that ring reflects the number of branch points of the covers parametrized by each component, and the multiplicative structure is induced by a geometric "concatenation" operation.

The definition of that ring is motivated by the fact that its Hilbert function is tightly related to the asymptotic behavior of the cohomology of Hurwitz spaces, which is in turn related (using the

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[1]Here, a marked G-cover is a finite branched cover (not necessarily connected) with a marked point in an unramified fiber, equipped with an action of G on the cover inducing simply transitive actions of G on each unramified fiber.

Grothendieck–Lefschetz trace formula) to the count of $\mathbb{F}_q$-points of Hurwitz spaces and hence to the distribution of field extensions of $\mathbb{F}_q(T)$ with Galois group isomorphic to G, when q is large and coprime to $|G|$. In [ETW17], this approach was used to obtain an upper bound consistent with the variant of Malle's conjecture for function fields over finite fields.

In [Seg24], we have extended some of the counting results of [EVW16]. For instance, we have studied the analogous ring of components of Hurwitz spaces of marked G-covers of the *projective* line. That ring is a *commutative* graded finitely generated algebra, and the growth of its Hilbert function is related to geometric invariants of its spectrum. This observation was the starting point for a more systematic study of the ring of components from the point of view of algebraic geometry.

1.2. Main results

In Section 2, we define the *ring of components* R (Definition 2.7), which is a finitely generated commutative graded k-algebra. We then introduce its prime spectrum $\operatorname{Spec} R$, which we call the *variety of components* (Definition 2.8). In Section 3, we define subsets $\gamma(H)$ of $\operatorname{Spec} R$ (Definition 3.4), indexed by subgroups H of G, and we prove that they form a stratification of the variety of components:

Theorem 1.1. *The locally closed subsets $\gamma(H)$ form a stratification of* $\operatorname{Spec} R$*:*

$$\operatorname{Spec} R = \bigsqcup_{H \subseteq G} \gamma(H).$$

This result, which is a particular case of the more general Theorem 3.9, has the following consequence: in order to describe the variety of components fully, it suffices to describe each stratum $\gamma(H)$. Using the counting results of [Seg24], we compute in Section 5 the Krull dimension of the stratum $\gamma(H)$ corresponding to a subgroup H of G. More precisely, we relate it to a numerical invariant defined in [Seg24], the *splitting number* $\Omega(D_H)$ (Definition 2.12):

Theorem 1.2. *We have* $\dim_{\mathrm{Krull}} \gamma(H) = \Omega(D_H) + 1$.

In Subsection 5.3, we discuss further connections between group-theoretic and geometric invariants by relating the degree of the stratum $\gamma(H)$, seen as embedded in projective space, to (a quotient of) the second homology group of H.

In Section 6, we approach the variety of components more directly by describing the strata in Theorem 6.16 and its coordinate-based variant Theorem 6.20. However, our description relies on strong assumptions on the ring of components. We do not reproduce the statement here as it uses a lot of terminology. This result applies in particular to the classical situation where G is a symmetric group and D contains only the conjugacy class of transpositions. In that case, Theorem 6.21 gives a full description of the variety of components and of its stratification.

1.3. Outline

This article is organized as follows:

- In Section 2, we define notation and terminology used throughout the article, notably the ring of components (Definition 2.7) and its associated variety (Definition 2.8) which are our main objects of study.
- In Section 3, we associate to each subgroup H of G a subring R^H and four ideals I_H, I_H^*, J_H, J_H^* of the ring of components (Definition 3.1). We use these to define the strata $\gamma(H)$ (Definition 3.4). We then prove Theorem 3.9, which is the general form of the stratification of the variety of components (Theorem 1.1).
- In Section 4, we prove Theorem 4.1. This technical result, which is a weak asymptotic form of reducedness for the ring of components R, is needed for the proof of Theorem 1.2.

- In Section 5, we compute the Krull dimension of each stratum $\gamma(H)$ (Theorem 1.2). In Subsection 5.3, we also compute the degree of $\gamma(H)$ in some cases. The proofs rely on the asymptotic counting results from [Seg24].
- In Section 6, we prove Theorems 6.16 and 6.20, which give complete descriptions of the variety of components in some cases. We apply these results to the classical case of symmetric groups in Subsection 6.5.

1.4. Acknowledgments

This work was funded by the French ministry of research through a CDSN grant, and by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) — Project-ID 491392403 — TRR 358. The author states that there is no conflict of interest.

I thank my advisors Pierre Dèbes and Ariane Mézard for their support and their precious advice during my time as a PhD student, and the reporters of my thesis Jean-Marc Couveignes and Craig Westerland for providing helpful feedback.

2. Definitions and preliminaries

Recall that we have fixed a finite group G, a set D of nontrivial conjugacy classes of G, and a map $\xi\colon D \to \mathbb{Z}_{>0}$. Additionally, we define the set $c \coloneqq \bigsqcup_{\gamma\in D} \gamma$ and the integer $|\xi| \coloneqq \sum_{\gamma\in D} \xi(\gamma)$.

2.1. The monoid of components

We briefly recall the definition of the *monoid of components* $\mathrm{Comp}_{\mathbb{P}^1(\mathbb{C})}(G, D, \xi)$ (cf. [Seg23, Definition 3.4.4] and [Seg24, Definition 2.6]). First, we define the braid group B_n by its presentation:

Definition 2.1. The *Artin braid group* B_n on n strands is defined by the following presentation:

$$\mathrm{B}_n \coloneqq \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ccccc} \sigma_i\sigma_j & = & \sigma_j\sigma_i & & \text{if } |i-j|>1 \\ \sigma_i\sigma_{i+1}\sigma_i & = & \sigma_{i+1}\sigma_i\sigma_{i+1} & & \text{if } i\in\{1,\ldots,n-2\} \end{array} \right\rangle.$$

Definition 2.2. The *Hurwitz action* of B_n on the set G^n of n-tuples of elements of G is the (well-defined) action for which the generator $\sigma_i \in \mathrm{B}_n$ acts on a tuple $\underline{g} = (g_1, \ldots, g_n) \in G^n$ as follows:

$$\sigma_i.(g_1, \ldots, g_{i-1}, g_i, g_{i+1}, g_{i+2}, \ldots, g_n) = (g_1, \ldots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \ldots, g_n).$$

Definition 2.3. Let $\underline{g} = (g_1, \ldots, g_n) \in G^n$ be a tuple of elements of G. The *group* of $\underline{g}$ is the subgroup $\left\langle \underline{g} \right\rangle$ of G generated by $g_1, \ldots, g_n$, and the *product* of $\underline{g}$ is the element $\pi\underline{g} \coloneqq g_1 \cdots g_n \in G$.

Both the group and product of a tuple are invariant under the Hurwitz action, so we extend the definition of these invariants and the notations $\langle m \rangle$ and πm when m is an orbit for the Hurwitz action.

Definition 2.4. A *component* (of degree n) is the orbit, under the Hurwitz action of the braid group $\mathrm{B}_{n|\xi|}$, of a tuple $\underline{g} = (g_1, \ldots, g_{n|\xi|}) \in G^{n|\xi|}$ satisfying $\pi\underline{g} = 1$ and such that exactly $n\cdot\xi(\gamma)$ entries of $\underline{g}$ belong to each conjugacy class $\gamma \in D$. The *monoid of components* $\mathrm{Comp}_{\mathbb{P}^1(\mathbb{C})}(G, D, \xi)$ is the (nonnegatively) graded set whose elements of degree n are the components of degree n, equipped with the multiplication induced by the concatenation of tuples:

$$(g_1, \ldots, g_{n|\xi|})(g'_1, \ldots, g'_{n'|\xi|}) = (g_1, \ldots, g_{n|\xi|}, g'_1, \ldots, g'_{n|\xi|}).$$

The monoid of components is well-defined and commutative [Seg23, Propositions 3.3.8 and 3.3.11]. Components of degree n are named this way because they correspond bijectively to connected components of the Hurwitz space classifying marked G-covers of the projective line branched at $n \cdot |\xi|$

points, among which $n \cdot \xi(\gamma)$ have their monodromy elements in each class $\gamma \in D$. (This connection is explained more carefully in [Seg23, Subsection 3.3.2].) The identity element of the monoid $\mathrm{Comp}_{\mathbb{P}^1(\mathbb{C})}(G, D, \xi)$ is the orbit of the empty tuple, which corresponds to the connected component containing only the trivial G-cover (with no branch points).

Definition 2.5. A nontrivial element of $\mathrm{Comp}_{\mathbb{P}^1(\mathbb{C})}(G, D, \xi)$ is a *non-factorizable component*[2] if it does not equal any product of two nontrivial components.

A simple pigeonhole argument (detailed in [Seg23, Lemma 3.4.17]) shows that there are finitely many non-factorizable components. Therefore, the monoid of components is a finitely generated commutative graded monoid.

Remark 2.6. The non-factorizable components do not necessarily all have the same degree; see [Seg23, Remark 3.4.20] for a counterexample.

2.2. The ring of components

2.2.1. Definition. We now define the ring and variety of components:

Definition 2.7. The *ring of components* R is the graded k-algebra $k[\mathrm{Comp}_{\mathbb{P}^1(\mathbb{C})}(G, D, \xi)]$ obtained as the monoid ring (over k) of the monoid of components. The *irrelevant ideal* ϖ is the (maximal) ideal of R generated by components of positive degree.

(The ring R corresponds to $R_{\mathbb{P}^1(\mathbb{C})}(G, D, \xi)$ in the notation of [Seg23, Definition 3.4.12].)

The properties stated in Subsection 2.1 imply that the ring R is a commutative graded k-algebra of finite type, generated by the non-factorizable components.

Definition 2.8. The *variety of components* is the set $\operatorname{Spec} R$ of prime ideals $\mathfrak{p} \subsetneq R$, equipped with the Zariski topology. If I is an ideal of R, we denote by $V(I)$ the closed subset of $\operatorname{Spec} R$ consisting of all prime ideals containing I.

2.2.2. Affine embedding. Assume that k is algebraically closed, and let $\operatorname{Spm} R$ be the subset of $\operatorname{Spec} R$ consisting of closed points, i.e., of maximal ideals $\mathfrak{m} \subsetneq R$. Then, the set $\operatorname{Spm} R$ can be identified with the set of morphisms of k-algebras from R to k (identifying a maximal ideal $\mathfrak{m}$ with the projection $R \twoheadrightarrow R/\mathfrak{m} \simeq k$), or equivalently with the set of k-points of the scheme $\operatorname{Spec} R$. Let Σ be the finite set of non-factorizable components. Then, we can identify $\operatorname{Spm} R$ with a classical variety by embedding it in the affine space k^Σ as follows: a point $(x_m)_{m\in\Sigma}$ belongs to $\operatorname{Spm} R$ if and only if the equality $x_{m_1} \cdots x_{m_u} = x_{m'_1} \cdots x_{m'_v}$ holds whenever the equality $m_1 \cdots m_u = m'_1 \cdots m'_v$ holds in the monoid of components. (Note that, as the monoid of components is commutative and finitely generated, Dickson's lemma implies that it is presented by finitely many equalities of that type.)

Remark 2.9. The ring R is a graded k-algebra, so it may be more natural to consider the projective variety that it defines (i.e., the set of homogeneous ideals which are maximal among those properly contained in ϖ) instead of the affine variety $\operatorname{Spm} R$. However, since non-factorizable components need not all have the same degree (cf. Remark 2.6), the space in which the variety naturally embeds is a weighted projective space, namely the set of orbits of $k^\Sigma \setminus \{0\}$ under the action of $k^\times$ for which a scalar $\lambda \in k^\times$ acts on a point z by multiplying its coordinate z_m (associated to a non-factorizable component $m \in \Sigma$) by $\lambda^{\deg m}$. Weighted projective spaces do embed in ordinary projective spaces of higher dimension [Hos20, Theorem 3.4.9], but we mostly work with the affine variety associated to R to avoid dealing with these subtleties.

[2] Non-factorizable components are simply the irreducible elements of the monoid $\mathrm{Comp}_{\mathbb{P}^1(\mathbb{C})}(G, D, \xi)$, but we avoid the ambiguous expression "irreducible component".

2.3. D-generated subgroups

We briefly recall the notion of *D-generated subgroups* (cf. [Seg24, Definition 1.1]):

Definition 2.10. A subgroup H of G is *D-generated* if the sets $\gamma \cap H$ for $\gamma \in D$ are all nonempty and collectively generate H. We denote by $\mathrm{Sub}_{G,D}$ the set of subgroups of G which are either trivial or D-generated.

The relevance of this definition comes from the following fact, proved in [Seg23, Proposition 3.2.22]:

Proposition 2.11. *A subgroup H of G belongs to $\mathrm{Sub}_{G,D}$ if and only if there is a component $m \in \mathrm{Comp}_{\mathbb{P}^1(\mathbb{C})}(G,\, D,\, \xi)$ whose group is H.*

If H is a D-generated subgroup of G, we define its *splitting number* as in [Seg24, Definition 1.2]:

Definition 2.12. Let $H \in \mathrm{Sub}_{G,D}$. Let $D_H := \{\gamma \cap H \,|\, \gamma \in D\}$, let $c_H := c \cap H$ (which is also $\bigsqcup_{\gamma' \in D_H} \gamma'$), and denote by D_H^* the set of conjugacy classes of H which are contained in c_H. The *splitting number* of H is the integer $\Omega(D_H) := |D_H^*| - |D_H|$.

Definition 2.13. A D-generated subgroup H of G is a *non-splitter* if $\Omega(D_H) = 0$, i.e., if D_H consists of conjugacy classes of H.

The splitting number of H plays a key role in the asymptotic count of components with group H, cf. [Seg24, Theorem 1.4].

2.4. Chart of notations

For quick reference, the chart below indicates where the definitions introduced in this section can be found. A short description is also given.

Notation	Reference	Short description
G, D, ξ, k	Top of Section 1	setup
$c, \lvert\xi\rvert$	Top of Section 2	
B_n	Definition 2.1	Artin braid group
$\langle \underline{g} \rangle, \pi\underline{g}$	Definition 2.3	group and product of a tuple (or component)
$\mathrm{Comp}_{\mathbb{P}^1(\mathbb{C})}(G,\, D,\, \xi)$	Definition 2.4	monoid of components
R	Definition 2.7	ring of components
$\mathrm{Spec}\, R, V(I)$	Definition 2.8	variety of components and its closed subsets
$\mathrm{Sub}_{G,D}$	Definition 2.10	set of D-generated (or trivial) subgroups
$\Omega(D_H)$	Definition 2.12	splitting number of H

We also include a chart of notation introduced in later sections:

Notation	Reference	Short description
I_H, I_H^*, J_H, J_H^*	Definition 3.1	ideals of R associated to a subgroup H
R^H	Definition 3.1	subring of R associated to a subgroup H
$\gamma(H)$	Definition 3.4	stratum associated to a subgroup H
Γ_H	Definition 3.6	ideal quotient $(\sqrt{I_H^*} : \sqrt{I_H})$
$R_{n,H}, N_{n,H}$	Top of Section 4	space spanned by components of degree n and group H (resp. subspace of nilpotent elements)
$\mu_H, H_2(H, c_H)$	Subsection 4.1	notation related to the lifting invariant

3. The subgroup stratification of the variety of components

In this section, we establish the stratification of the variety of components (Theorem 1.1). To this end, we prove the more general Theorem 3.9.

3.1. Remarkable ideals and subrings of the ring of components

Definition 3.1. Let H be a subgroup of G. We define the following graded k-linear subspaces of the ring of components R, each of which is the space spanned by components $m \in \mathrm{Comp}_{\mathbb{P}^1(\mathbb{C})}(G, D, \xi)$ (Definition 2.4) whose group $\langle m \rangle$ (Definition 2.3) satisfies some condition:

- I_H is spanned by components whose group contains H.
- I_H^* is spanned by components whose group properly contains H.
- J_H is spanned by components whose group is not properly contained in H.
- J_H^* is spanned by components whose group is not contained in H.
- R^H is spanned by components whose group is contained in H.

For any subgroup $H \subseteq G$, the subspaces I_H, I_H^*, J_H and J_H^* are homogeneous ideals of R, and R^H is a graded subalgebra of R, isomorphic to the quotient R/J_H^* (cf. [Seg23, Proposition 3.4.26]). Note that, by Proposition 2.11, the inclusions $I_H^* \subseteq I_H$ and $J_H^* \subseteq J_H$ are strict if and only if $H \in \mathrm{Sub}_{G,D}$. The following properties are straightforward to check (cf. [Seg23, Proposition 3.4.24]):

Proposition 3.2. *The subspaces defined in Definition 3.1 satisfy the following properties:*

(i) Let $H \subseteq H'$ be subgroups of G. Then, we have the inclusion $R^H \subseteq R^{H'}$ between the associated subalgebras of R, and "mirrored" inclusions between the associated ideals of R:

$$I_{H'} \subseteq I_H \qquad I_{H'}^* \subseteq I_H^* \qquad J_{H'} \subseteq J_H \qquad J_{H'}^* \subseteq J_H^*.$$

(ii) We have:

$$I_1 = J_1 = R^G = R \qquad I_1^* = J_1^* = \varpi \qquad I_G^* = J_G^* = 0 \qquad R^1 = k.$$

(iii) For every subgroup $H \subseteq G$, we have:

$$I_H^* = I_H \cap J_H^* \qquad J_H = I_H + J_H^*.$$

(iv) For every subgroup $H \subseteq G$, we have:

$$I_H^* = \sum_{H' \supsetneq H} I_{H'} \qquad J_H = \bigcap_{H' \subsetneq H} J_{H'}^*.$$

(v) For all subgroups $H_1, H_2 \subseteq G$, we have:

$$I_{\langle H_1, H_2 \rangle} = I_{H_1} \cap I_{H_2} \qquad J_{H_1 \cap H_2}^* = J_{H_1}^* + J_{H_2}^*.$$

We also rephrase some of the properties from Proposition 3.2 geometrically:

Proposition 3.3. *The closed subsets $V(I_H)$, $V(I_H^*)$, $V(J_H)$ and $V(J_H^*)$ of $\operatorname{Spec} R$ associated to the ideals from Definition 3.1 satisfy the following properties:*

(i) Let $H \subseteq H'$ be subgroups of G. Then, the following inclusions hold:

$$V(I_H) \subseteq V(I_{H'}) \qquad V(I_H^*) \subseteq V(I_{H'}^*) \qquad V(J_H) \subseteq V(J_{H'}) \qquad V(J_H^*) \subseteq V(J_{H'}^*).$$

(ii) We have:

$$V(I_1) = V(J_1) = \varnothing \qquad V(I_1^*) = V(J_1^*) = \{\varpi\} \qquad V(I_G^*) = V(J_G^*) = \operatorname{Spec} R.$$

(iii) For every subgroup $H \subseteq G$, we have:

$$V(I_H^*) = V(I_H) \cup V(J_H^*) \qquad V(J_H) = V(I_H) \cap V(J_H^*).$$

(iv) For every subgroup $H \subseteq G$, we have:

$$V(I_H^*) = \bigcap_{H' \supsetneq H} V(I_{H'}) \qquad V(J_H) = \bigcup_{H' \subsetneq H} V(J_{H'}^*).$$

(v) For all subgroups $H_1, H_2 \subseteq G$, we have:

$$V(I_{\langle H_1, H_2 \rangle}) = V(I_{H_1}) \cup V(I_{H_2}) \qquad V(J_{H_1 \cap H_2}^*) = V(J_{H_1}^*) \cap V(J_{H_2}^*).$$

3.2. The strata $\gamma(H)$

We fix a subgroup H of G.

Definition 3.4. The *stratum associated to* H is the (locally closed) subset $\gamma(H) \coloneqq V(I_H^*) \setminus V(I_H)$ of $\operatorname{Spec} R$, i.e., the set of prime ideals of R which contain I_H^* but do not contain I_H.

In other words, a prime ideal p of R belongs to $\gamma(H)$ if and only if p contains every component whose group is strictly larger than H, but there is a component m of group exactly H which does not belong to p. The stratum $\gamma(H)$ is nonempty if and only if $H \in \operatorname{Sub}_{G,D}$. Note that $\gamma(1) = \{\varpi\}$ by Proposition 3.3 (ii).

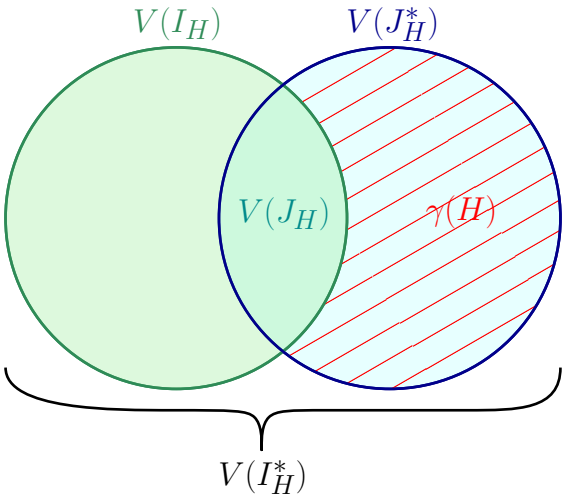
Proposition 3.5. *The stratum $\gamma(H)$ is equal to $V(J_H^*) \setminus V(J_H)$.*

Proof.

$$\begin{aligned}
\gamma(H) &= V(I_H^*) \setminus V(I_H) && \text{by Definition 3.4} \\
&= \left(V(I_H) \cup V(J_H^*)\right) \setminus V(I_H) && \text{by Proposition 3.3 (iii)} \\
&= V(J_H^*) \setminus \left(V(I_H) \cap V(J_H^*)\right) \\
&= V(J_H^*) \setminus V(J_H) && \text{by Proposition 3.3 (iii).}
\end{aligned}$$

□

We summarize the relations between $V(I_H^*)$, $V(I_H)$, $V(J_H^*)$, $V(J_H)$ and $\gamma(H)$ in a diagram:



The closure of the strata in $\operatorname{Spec} R$ can be described using ideal quotients:

Definition 3.6. We define the subset $\Gamma_H \subseteq R$ as the ideal quotient $(\sqrt{I_H^*} : \sqrt{I_H})$:

$$\Gamma_H \coloneqq (\sqrt{I_H^*} : \sqrt{I_H}) = \left\{ x \in R \,\middle|\, x\sqrt{I_H} \subseteq \sqrt{I_H^*} \right\} = \left\{ x \in R \,\middle|\, (x) \cap \sqrt{I_H} \subseteq \sqrt{I_H^*} \right\}.$$

(The last equality comes from the fact that a product of two ideals has the same radical as their intersection.)

Proposition 3.7. *The set Γ_H satisfies the following properties:*

(i) Γ_H *is a homogeneous radical ideal of* R*, namely the intersection of all prime ideals* $p \in \gamma(H)$*.*

(ii) The closed subset $V(\Gamma_H) \subseteq \operatorname{Spec} R$ *associated to* Γ_H *is the closure of* $V(I_H) \setminus V(I_H^*) = \gamma(H)$*.*

(iii) $\dim_{\mathrm{Krull}} R/\Gamma_H = \dim_{\mathrm{Krull}} \overline{\gamma(H)} = \dim_{\mathrm{Krull}} \gamma(H)$*.*

(iv) $\Gamma_H \cap \sqrt{I_H} = \sqrt{I_H^*}$*.*

(v) An ideal I *of* R *satisfies* $I \cap \sqrt{I_H} \subseteq \sqrt{I_H^*}$ *if and only if* I *is contained in* Γ_H*.*

(vi) Any ideal I *of* R *satisfying* $I \cap I_H \subseteq \Gamma_H$ *is contained in* Γ_H*.*

Proof. Points (i) and (ii) follow from well-known properties of ideal quotients, see for example [CLO15, Chapter 4, §4]. Point (iii) follows from (ii) (for the first equality) and [Har77, Proposition I.1.10] (for the second equality). Points (iv) and (v) follow directly from the definition of Γ_H. For point (vi), assume that I is an ideal such that $I \cap I_H \subseteq \Gamma_H$. Then, $I \cap \sqrt{I_H} \subseteq \sqrt{I \cap I_H} \subseteq \Gamma_H \cap \sqrt{I_H} = \sqrt{I_H^*}$, where the last equality is point (iv). By point (v), we then have $I \subseteq \Gamma_H$. □

3.3. The subgroup stratification of $\operatorname{Spec} R$

In this subsection, we show that the strata $\gamma(H)$ defined in Definition 3.4 form a stratification of the variety of components. We first show that the strata are disjoint:

Proposition 3.8. *Let* H*,* H' *be two distinct subgroups of* G*. Then* $\gamma(H) \cap \gamma(H') = \varnothing$*.*

Proof. If $H \subsetneq H'$, then the set $\gamma(H)$ is contained in $V(I_H^*)$ and hence in $V(I_{H'})$ by Proposition 3.3 (iv), and thus $\gamma(H')$ has an empty intersection with $\gamma(H)$. The case $H' \subsetneq H$ is symmetric. We now assume that neither of H and H' is contained in the other, so that $\widetilde{H} = \langle H, H' \rangle$ is a subgroup of G strictly larger than both H and H'. We let $J \coloneqq \gamma(H) \cap \gamma(H')$. By Proposition 3.3 (iv), we have $V(I_H^*) \subseteq V(I_{\widetilde{H}})$ and hence $\gamma(H) \subseteq V(I_{\widetilde{H}})$. The same holds for H' and thus $J \subseteq V(I_{\widetilde{H}})$. But then $V(I_{\widetilde{H}}) \setminus J$ is a subset of $V(I_{\widetilde{H}})$ containing both $V(I_H)$ and $V(I_{H'})$. By Proposition 3.3 (v), we have $V(I_{\widetilde{H}}) = V(I_H) \cup V(I_{H'})$ which finally implies $J = \varnothing$. □

We now prove the general form of the stratification:

Theorem 3.9. *For every subgroup* $H \subseteq G$*, we have the following equalities:*

$$V(I_H) = \bigsqcup_{\substack{H' \text{ subgroup of } G \\ \text{not containing } H}} \gamma(H') \qquad V(I_H^*) = \bigsqcup_{\substack{H' \text{ subgroup of } G \\ \text{not properly containing } H}} \gamma(H')$$

$$V(J_H) = \bigsqcup_{H' \subsetneq H} \gamma(H') \qquad V(J_H^*) = \bigsqcup_{H' \subseteq H} \gamma(H').$$

By setting $H = G$ in the equality concerning $V(I_H^*)$, Theorem 3.9 implies Theorem 1.1.

Proof. We show by decreasing induction on the size of a subgroup H of G that:

$$\operatorname{Spec} R \setminus V(I_H) = \bigcup_{H' \supseteq H} \gamma(H'). \tag{3.1}$$

Let H be a subgroup of G, and assume that every subgroup $H' \subseteq G$ with $|H'| > |H|$ satisfies Equation (3.1). Then:

$$\begin{aligned}
\operatorname{Spec} R \setminus V(I_H^*) &= \operatorname{Spec} R \setminus \bigcap_{H' \supsetneq H} V(I_{H'}) && \text{by Proposition 3.3 (iv)} \\
&= \bigcup_{H' \supsetneq H} \Big(\operatorname{Spec} R \setminus V(I_{H'}) \Big) \\
&= \bigcup_{H' \supsetneq H} \bigcup_{H'' \supseteq H'} \gamma(H'') && \text{by the induction hypothesis} \\
&= \bigcup_{H' \supsetneq H} \gamma(H').
\end{aligned}$$

We conclude the induction by using the equality $\operatorname{Spec} R \setminus V(I_H) = \gamma(H) \cup (\operatorname{Spec} R \setminus V(I_H^*))$, which follows directly from Definition 3.4. The union in Equation (3.1) is a disjoint union by Proposition 3.8. Plugging in $H = 1$ yields:

$$\operatorname{Spec} R = \operatorname{Spec} R \setminus V(I_1) = \bigsqcup_{H' \subseteq G} \gamma(H') \tag{3.2}$$

which is Theorem 1.1. Finally:

$$\begin{aligned}
V(I_H) &= \operatorname{Spec} R \setminus \Big(\operatorname{Spec} R \setminus V(I_H) \Big) \\
&= \left(\bigsqcup_{H' \subseteq G} \gamma(H') \right) \setminus \left(\bigsqcup_{H' \supseteq H} \gamma(H') \right) && \text{by Equations (3.1) and (3.2)} \\
&= \bigsqcup_{\substack{H' \text{ subgroup of } G \\ \text{not containing } H}} \gamma(H').
\end{aligned}$$

Since $V(I_H^*) = V(I_H) \cup \gamma(H)$ (Definition 3.4), we have the corresponding formula for $V(I_H^*)$.

Dually, we show by increasing induction on the size of a subgroup $H \subseteq G$ that:

$$V(J_H^*) = \bigcup_{H' \subseteq H} \gamma(H'). \tag{3.3}$$

Let H be a subgroup of G, and assume that every subgroup H' of G with $|H'| < |H|$ satisfies Equation (3.3). Then:

$$\begin{aligned}
V(J_H) &= \bigcup_{H' \subsetneq H} V(J_{H'}^*) && \text{by Proposition 3.3 (iv)} \\
&= \bigcup_{H' \subsetneq H} \bigcup_{H'' \subseteq H'} \gamma(H'') && \text{by the induction hypothesis} \\
&= \bigcup_{H' \subsetneq H} \gamma(H').
\end{aligned}$$

We conclude by using Proposition 3.5 to relate $V(J_H)$ and $V(J_H^*) = \gamma(H) \cup V(J_H)$. □

4. Nilpotent elements of the ring of components

In this section, we prove Theorem 4.1, which states that the ring of components satisfies a weak asymptotic form of reducedness. This property is required for the proof of Theorem 1.2.

For the whole section, fix a subgroup H of G, let $c_H \coloneqq c \cap H = \bigsqcup_{\gamma \in D} \gamma \cap H$, and let D_H^* be the set of all conjugacy classes of H contained in c_H. Recall from Definition 2.12 that the splitting number of H is $\Omega(D_H) = |D_H^*| - |D|$. For each $n \in \mathbb{N}$, let $R_{n,H}$ be the linear subspace of R spanned by components of group exactly H and of degree n, and let $N_{n,H}$ be the space of nilpotent elements of $R_{n,H}$. In other words, $R_{n,H} = (I_H \cap R^H)_n$ and $N_{n,H} = \sqrt{0} \cap R_{n,H}$. We now state the theorem:

Theorem 4.1. $\dim_k N_{n,H} = O\Big(n^{\Omega(D_H)-1}\Big)$.

This should be compared to [Seg24, Proposition 3.5 (ii)], which implies that $\dim_k R_{n,H} \neq o\Big(n^{\Omega(D_H)}\Big)$. Therefore, in some sense, most high-degree elements of R are not nilpotent. The main ingredient of the proof is Lemma 4.6, a result about the monoid of components whose proof involves the lifting invariant from [EVW12, Woo21] (briefly reviewed in Subsection 4.1). The case where k has positive characteristic (coprime to $|G|$) is Corollary 4.7. The case of characteristic 0 is Corollary 4.8, and follows from the case of positive characteristic using classical results from model theory.

4.1. Quick review of the lifting invariant

In this short subsection, we recall some properties of the lifting invariant from [EVW12, Woo21].

Definition 4.2. If $m \in \mathrm{Comp}_{\mathbb{P}^1(\mathbb{C})}(G,\, D,\, \xi)$ is a component of group contained in H, represented by a tuple $\underline{g}$ of elements of G, its *H-multidiscriminant* is the map $\mu_H(m)\colon D_H^* \to \mathbb{Z}$ mapping a conjugacy class $\gamma \in D_H^*$ to the number of entries of $\underline{g}$ which belong to γ. (This is independent of $\underline{g}$.)

Definition 4.3. If M is an integer and $m \in \mathrm{Comp}_{\mathbb{P}^1(\mathbb{C})}(G,\, D,\, \xi)$ is a component of group H, we say that m is *M-big* if its H-multidiscriminant $\mu_H(m)$ satisfies $\mu_H(m)(\gamma) \geq M$ for each class $\gamma \in D_H^*$.

In [Woo21], a specific quotient $H_2(H, c_H)$ of the homology group $H_2(H, \mathbb{Z})$ is defined. We have a monoid homomorphism $\mathrm{Comp}_{\mathbb{P}^1(\mathbb{C})}(G,\, D,\, \xi) \to H_2(H, c_H)$ (the first coordinate of the lifting invariant), cf. [Seg24, Subsection 4.1]. The key properties of this invariant are summarized in the following theorem [Woo21, Theorem 3.1 and Theorem 2.5] (see also [Seg24, Theorems 4.1 and 4.5, Remark 4.3]):

Theorem 4.4. *There exists a constant M, depending only on the group G, such that any two M-big components of group H with equal H-multidiscriminants and equal images in $H_2(H, c_H)$ are equal.*

Fix an integer M as in Theorem 4.4. We conclude this subsection by recalling the lemma [Seg24, Lemma 4.6], which implies that "most components are M-big":

Lemma 4.5. *For $n \in \mathbb{N}$, denote by $R_{n,H}^{M\text{-small}}$ the subspace of R spanned by components of group H and degree n which are not M-big. Then, $\dim_k R_{n,H}^{M\text{-small}} = O\Big(n^{\Omega(D_H)-1}\Big)$.*

4.2. Torsion in the monoid of components

We use the notation from Subsection 4.1. In particular, we fix a constant M as in Theorem 4.4. We prove the following lemma, which implies that the monoid of components is "not far" from being torsionfree (a monoid is torsionfree if $x^r = y^r$ for $r \geq 1$ implies $x = y$):

Lemma 4.6. *Let x and y be two M-big components of group H such that, for some integer $r \in \mathbb{N}$ coprime with $|H|$, we have $x^r = y^r$. Then, $x = y$.*

Proof. Since $x^r = y^r$, we have $\mu_H(x) = \frac{1}{r} \cdot \mu_H(x^r) = \frac{1}{r} \cdot \mu_H(y^r) = \mu_H(y)$. By Theorem 4.4, it remains only to show that the respective images x' and y' of x and y in $H_2(H, c_H)$ are equal. The equality $x^r = y^r$ implies $(x')^r = (y')^r$, which in turn implies that $x'(y')^{-1}$ is an element of r-torsion of the abelian group $H_2(H, c_H)$. However, the group $H_2(H, c_H)$, as a quotient of the second homology group $H_2(H, \mathbb{Z})$, is of $|H|$-torsion (cf. [Ser62, Chap. VIII, §2, Cor. 1]) and hence does not have any nontrivial r-torsion (r is coprime with $|H|$). We have shown that $x'(y')^{-1} = 1$. □

4.3. The case of positive characteristic

We fix a constant M as in Theorem 4.4. Using Lemma 4.5, we also fix a constant $\widetilde{C}$ such that, for all $n \in \mathbb{N}$, the number of components $m \in \mathrm{Comp}_{\mathbb{P}^1(\mathbb{C})}(G, D, \xi)$ of group H and degree n which are not M-big is bounded above by $\widetilde{C} \cdot n^{\Omega(D_H)-1}$.

Corollary 4.7. *Assume that the characteristic p of k is positive and coprime to $|G|$. Then:*

$$\forall n \in \mathbb{N}, \quad \dim_k N_{n,H} \leq \widetilde{C} \cdot n^{\Omega(D_H)-1}.$$

Proof. Fix some $n \in \mathbb{N}$. Let u^r be the k-linear map $R_{n,H} \to R_{np^r,H}$ induced by $x \mapsto x^{p^r}$ on components. It follows from Lemma 4.6 and Lemma 4.5 that $\dim_k \ker(u^r) \leq \widetilde{C} \cdot n^{\Omega(D_H)-1}$ for all $r \in \mathbb{N}$.

Choose a basis $x_1, \ldots, x_D$ of $N_{n,H}$, and complete it into a basis $\underbrace{x_1, \ldots, x_D}_{\in N_{n,H}}, x_{D+1}, \ldots, x_{D'}$ of $R_{n,H}$. Note that $D = \dim N_{n,H}$ and $D' = \dim R_{n,H}$. Express the vectors $x_1, \ldots, x_{D'}$ in the basis $m_1, \ldots, m_{D'}$ of $R_{n,H}$ given by all components of group H and of degree n:

$$x_i = \sum_{j=1}^{D'} \lambda_{i,j} m_j.$$

Since $x_1, \ldots, x_D$ are nilpotent, fix $r \geq 1$ such that $x_i^{p^r} = 0$ for all $i \in \{1, \ldots, D\}$. For $i \in \{1, \ldots, D'\}$, define:

$$\widetilde{x}_i \coloneqq \sum_{j=1}^{D'} \lambda_{i,j}^{p^r} m_j.$$

Then, $(\widetilde{x}_1, \ldots, \widetilde{x}_{D'})$ is still a basis of $R_{n,H}$. Indeed, by the properties of the Frobenius endomorphism, the determinant of the matrix $(\lambda_{i,j}^{p^r})_{i,j}$ is the p^r-th power of the determinant of the matrix $(\lambda_{i,j})_{i,j}$, which is nonzero because $x_1, \ldots, x_{D'}$ is a basis. In particular, the vectors $\widetilde{x}_1, \ldots, \widetilde{x}_D$ are linearly independent.

Now, if $i \in \{1, \ldots, D\}$, we have:

$$u^r(\widetilde{x}_i) = \sum_{j=1}^{D'} \lambda_{i,j}^{p^r} u^r(m_j) = \sum_{j=1}^{D'} \lambda_{i,j}^{p^r} m_j^{p^r} = \left(\sum_{j=1}^{D'} \lambda_{i,j} m_j \right)^{p^r} = x_i^{p^r} = 0.$$

So $\widetilde{x}_1, \ldots, \widetilde{x}_D$ is a list of D linearly independent vectors in $\ker(u^r)$. The existence of such a list implies that $D \leq \dim_k \ker(u^r)$, and finally $\dim_k N_{n,H} = D \leq \dim_k \ker(u^r) \leq \widetilde{C} \cdot n^{\Omega(D_H)-1}$. □

4.4. The case of characteristic zero

The setting and the notation are the same as in Subsection 4.3.

Corollary 4.8. *Assume that the characteristic of k is zero. Then:*

$$\forall n \in \mathbb{N}, \quad \dim_k N_{n,H} \leq \widetilde{C} \cdot n^{\Omega(D_H)-1}.$$

Proof. Since dimensions do not change under field extensions, we may assume that k is algebraically closed. Fix an integer $n \in \mathbb{N}$ and denote by F the finite set of components of degree n and group H. For each $r \geq 1$, define the following finite subset of $\mathrm{Comp}_{\mathbb{P}^1(\mathbb{C})}(G, D, \xi)$:

$$F^{\wedge r} = \{m_1 m_2 \cdots m_r \,|\, m_1, m_2, \ldots, m_r \in F\}.$$

Consider an arbitrary field K. For every $r \geq 1$ and every element $x = \sum_{m \in F} \lambda_m m$ in the K-vector space spanned by F, we have the following equality in $K[\mathrm{Comp}_{\mathbb{P}^1(\mathbb{C})}(G, D, \xi)]$:

$$\begin{aligned} x^r &= \sum_{m_1, m_2, \ldots, m_r \in F} \lambda_{m_1} \lambda_{m_2} \cdots \lambda_{m_r} \cdot (m_1 m_2 \cdots m_r) \\ &= \sum_{m \in F^{\wedge r}} \left(\sum_{\substack{m_1, m_2, \ldots, m_r \in F \\ m_1 m_2 \cdots m_r = m}} \lambda_{m_1} \lambda_{m_2} \cdots \lambda_{m_r} \right) m. \end{aligned}$$

If $\underline{\lambda} = (\lambda_m)_{m\in F}$ is a tuple of variables indexed by F, we denote by $\mathtt{Nilp}_r(\underline{\lambda})$ the following conjunction, which is a first-order property in the language of fields expressing the fact that the element $x = \sum_m \lambda_m m$ satisfies $x^r = 0$ in $K[\mathrm{Comp}_{\mathbb{P}^1(\mathbb{C})}(G, D, \xi)]$:

$$\bigwedge_{m\in F^{\wedge r}} \left(\sum_{\substack{m_1,m_2,\ldots,m_r\in F \\ m_1m_2\cdots m_r = m}} \lambda_{m_1}\lambda_{m_2}\cdots\lambda_{m_r} = 0 \right).$$

Let $d := \widetilde{C}\cdot n^{\Omega(D_H)-1}+1$. If $\underline{\lambda}^{(1)},\ldots,\underline{\lambda}^{(d)}$ are d tuples of variables, each indexed by F, we denote by $\mathtt{Indep}(\underline{\lambda}^{(1)},\ldots,\underline{\lambda}^{(d)})$ the following first-order property in the language of fields, expressing the fact that the elements $x^{(i)} = \sum_m \lambda_m^{(i)} m$ are linearly independent:

$$\forall x_1,\ldots,\forall x_d,\quad \left(\bigwedge_{m\in F}[x_1\lambda_m^{(1)}+\cdots+x_d\lambda_m^{(d)}=0]\right) \implies (x_1=0\wedge x_2=0\wedge\ldots\wedge x_d=0).$$

Finally, we define the first-order property φ_r as:

$$\forall\underline{\lambda}^{(1)},\ldots,\forall\underline{\lambda}^{(d)},\quad \left(\mathtt{Nilp}_r(\underline{\lambda}^{(1)})\wedge\ldots\wedge\mathtt{Nilp}_r(\underline{\lambda}^{(d)})\right) \implies \neg\mathtt{Indep}(\underline{\lambda}^{(1)},\ldots,\underline{\lambda}^{(d)}),$$

expressing the fact that any elements $x^{(1)},\ldots,x^{(d)} \in \mathrm{Span}_K(F)$ whose r-th powers all vanish in $K[\mathrm{Comp}_{\mathbb{P}^1(\mathbb{C})}(G, D, \xi)]$ are linearly dependent, i.e., that the dimension of the subspace of $\mathrm{Span}_K(F)$ spanned by elements whose r-th powers are zero is at most $d-1 = \widetilde{C}\cdot n^{\Omega(D_H)-1}$.

By Corollary 4.7, the field $\overline{\mathbb{F}}_p$ satisfies the property φ_r for all $r\in\mathbb{N}$ when p is a prime not dividing $|G|$. Let $\mathcal{U}$ be a non-principal ultrafilter on the set $\mathcal{P}$ of primes coprime to $|G|$, and let $\mathbb{K}$ be the ultraproduct $\prod_{p\in\mathcal{P}}\overline{\mathbb{F}_p}/\mathcal{U}$, i.e., the quotient of the ring $\prod_{p\in\mathcal{P}}\overline{\mathbb{F}_p}$ by the (maximal) ideal generated by elements whose coordinates vanish for all primes p in some set $\mathcal{P}'\in\mathcal{U}$. By Łoś's theorem [Mar02, Exercise 2.5.18], $\mathbb{K}$ is an algebraically closed field of characteristic zero satisfying φ_r for all $r\geq 1$. Since the theory of algebraically closed fields of characteristic zero is complete [Mar02, Corollary 3.2.3], the same is true of k. Therefore, the dimension of the subspace $N_{n,H,r}$ of $R_{n,H}$ spanned by elements whose r-th power vanishes is at most $\widetilde{C}\cdot n^{\Omega(D_H)-1}$, for all $r\geq 1$. As the space $N_{n,H}$ is the increasing union of the subspaces $N_{n,H,r}$ for $r\geq 1$, we have:

$$\dim_k N_{n,H} = \sup_{r\geq 1}(\dim_k N_{n,H,r}) \leq \widetilde{C}\cdot n^{\Omega(D_H)-1}. \qquad \square$$

5. Dimensions and degrees of the strata

In this section, we prove Theorem 1.2. We fix a nontrivial D-generated subgroup $H\in\mathrm{Sub}_{G,D}$, and we prove that the Krull dimension of the stratum $\gamma(H)$ (Definition 3.4) is related to the splitting number $\Omega(D_H)$ (Definition 2.12) by the equality $\dim_{\mathrm{Krull}}\gamma(H) = \Omega(D_H)+1$.

The section is organized in the following way:

- In Subsection 5.1, we introduce tools we need. We prove a bound on the Krull dimension of some quotients (Lemma 5.1), we recall the general form of the Hilbert–Serre theorem for algebras whose generators have unequal degrees (Theorem 5.2), and we combine these two ingredients in Corollary 5.3.
- In Subsection 5.2, we compute the Krull dimension of the strata (Theorem 1.2) using the tools from Subsection 5.1 and the counting results from [Seg24].
- In Subsection 5.3, we relate the degree of $\gamma(H)$ (embedded in projective space) to the leading coefficients computed in [Seg24, Section 4].

5.1. Preliminaries

In this subsection, we introduce tools which are later used to compute the dimension of $\gamma(H)$. The main tool is the following lemma:

Lemma 5.1. *Let B be a nonzero finitely generated commutative graded k-algebra and J be a homogeneous ideal of B such that, for all ideals I of B, we have:*

$$I \cap J \subseteq \sqrt{0} \implies I \subseteq \sqrt{0}. \tag{5.1}$$

Then, we have $\dim_{\text{Krull}}(B/J) < \dim_{\text{Krull}} B$.

Proof. Let $p_1, \dots, p_u$ be the minimal homogeneous prime ideals of B, whose number is finite as B is Noetherian.

Assume by contradiction that J is contained in p_i for some $i \in \{1, \dots, u\}$, and let $V_i \coloneqq \bigcap_{j \neq i} p_j$. The ideal $V_i \cap p_i$ is the intersection of all minimal homogeneous primes of B, i.e., the nilradical $\sqrt{0}$. Since $J \subseteq p_i$, we have $V_i \cap J \subseteq V_i \cap p_i = \sqrt{0}$. By Equation (5.1), this implies $V_i \subseteq \sqrt{0}$. But then $V_i \subseteq p_i$ with p_i prime, and thus some p_j with $j \neq i$ must be contained in p_i, contradicting the minimality of p_i. We have shown that J is not contained in any p_i.

Consider a maximal chain $J \subseteq q_0 \subsetneq q_1 \subsetneq \dots \subsetneq q_{\dim_{\text{Krull}}(B/J)}$ of prime ideals of B containing J. By the previous paragraph, the inclusion $J \subseteq q_0$ implies that q_0 is not a minimal homogeneous prime ideal of B. Therefore, q_0 properly contains some p_i for $i \in \{1, \dots, u\}$. We then have the chain $p_i \subsetneq q_0 \subsetneq q_1 \subsetneq \dots \subsetneq q_{\dim_{\text{Krull}}(B/J)}$ of homogeneous prime ideals of B, proving that $\dim_{\text{Krull}} B > \dim_{\text{Krull}}(B/J)$. □

A second tool is Theorem 5.2, which is a variant of the Hilbert–Serre theorem for graded algebras whose generators are not necessarily of degree 1. The case where all generators have degree 1 is classical (see for example [Har77, Theorem I.7.5]), and in that case the Hilbert function eventually coincides with a polynomial. In the general case, there is instead a finite list of polynomials through which the Hilbert function eventually cycles periodically: we have a *Hilbert quasi-polynomial*.

Theorem 5.2 (Hilbert-Serre)**.** *Let A be a commutative graded k-algebra of positive Krull dimension, generated by finitely many generators of respective degrees $d_1, \dots, d_N$. Let $W \coloneqq \operatorname{lcm}(d_1, \dots, d_N)$. Then, there exist (uniquely defined) polynomials $Q_0, \dots, Q_{W-1}$ such that:*

(i) For all $m \in \{0, \dots, W-1\}$ and n large enough, we have $\mathrm{HF}_A(Wn+m) = Q_m(n)$;

(ii) $\deg Q_0 = \dim_{\text{Krull}}(A) - 1$*;*

(iii) $\deg Q_m \leq \dim_{\text{Krull}}(A) - 1$ *for all $m \in \{1, \dots, W-1\}$.*

For proofs, we refer the reader to [BH98, Theorem 4.4.3] or [Seg23, Lemmas 5.4.4, 5.4.5 and 5.4.6].

To say that two functions f and g satisfy $f = O(g)$ and $f \neq o(g)$, we write $f = O^\sharp(g)$. With this notation, Theorem 5.2 implies the following, for any algebra A satisfying the hypotheses:

$$\mathrm{HF}_A(n) = O^\sharp\Big(n^{\dim_{\text{Krull}}(A)-1}\Big). \tag{5.2}$$

Together, Theorem 5.2 and Lemma 5.1 directly imply:

Corollary 5.3. *Under the hypotheses of Lemma 5.1, we have* $\dim_k J_n = \mathrm{HF}_B(n) + O\Big(n^{\dim_{\text{Krull}}(B)-2}\Big)$.

5.2. The dimension of $\gamma(H)$

We are ready to prove Theorem 1.2 : the Krull dimension of the stratum $\gamma(H)$ is one more than the splitting number $\Omega(D_H)$.

Proof of Theorem 1.2. Let $R_{n,H} \coloneqq (I_H \cap R^H)_n$ be the space spanned by components of group H and of degree n. In order to apply Corollary 5.3, we check that the hypotheses of Lemma 5.1 hold when $B = R^H/(\Gamma_H \cap R^H)$ and J is the image of $I_H \cap R^H$ in B. First, B is a finitely generated commutative graded k-algebra (cf. Subsection 2.2) and J is a homogeneous ideal of B. Since the ideal Γ_H being radical, B is reduced and hence $\sqrt{0} = 0$ in B. We consider an ideal I of B such that $I \cap J = 0$, and we want to show that $I = 0$. Let $\widetilde{I}$ be the inverse image of I in R^H. Then, $I \cap J = 0$ rewrites as $\widetilde{I} \cap I_H \subseteq \Gamma_H \cap R^H$. By Proposition 3.7 (vi), this implies $\widetilde{I} \subseteq \Gamma_H$, i.e., $I = 0$. We have checked the hypotheses of Lemma 5.1. We obtain:

$$\begin{aligned}
\dim_k J_n &= \mathrm{HF}_{R^H/(\Gamma_H \cap R^H)}(n) + O\Big(n^{\dim_{\mathrm{Krull}}(R^H/(\Gamma_H \cap R^H))-2}\Big) && \text{by Corollary 5.3} \\
&= O^\sharp\Big(n^{\dim_{\mathrm{Krull}}(R^H/(\Gamma_H \cap R^H))-1}\Big) && \text{by Equation (5.2)} \\
&= O^\sharp\Big(n^{\dim_{\mathrm{Krull}}\gamma(H)-1}\Big) && \text{by } \begin{cases} R^H/(\Gamma_H \cap R^H) \simeq R/\Gamma_H \\ \text{Proposition 3.7 (iii)} \end{cases}
\end{aligned}$$

By Proposition 3.7 (iv), the intersection $\Gamma_H \cap I_H$ is contained in $\sqrt{I_H^*}$. As one cannot obtain nonzero terms involving components of group strictly larger than H by taking powers of linear combinations of components of group exactly H, elements of $\sqrt{I_H^*} \cap I_H \cap R^H$ are exactly the nilpotent elements of $I_H \cap R^H$. Thus, $(\Gamma_H \cap I_H \cap R^H)_n \subseteq (\sqrt{I_H^*} \cap I_H \cap R^H)_n = (\sqrt{0} \cap I_H \cap R^H)_n = N_{n,H}$. We have:

$$\begin{aligned}
\dim_k J_n &= \dim_k (I_H \cap R^H)_n - \dim_k (\Gamma_H \cap I_H \cap R^H)_n && \text{by definition of } J \\
&= \dim_k R_{n,H} + O(\dim_k N_{n,H}) && \text{as } \begin{cases} (I_H \cap R^H)_n = R_{n,H} \\ (\Gamma_H \cap I_H \cap R^H)_n \subseteq N_{n,H} \end{cases} \\
&= O^\sharp\Big(n^{\Omega(D_H)}\Big) + O\Big(n^{\Omega(D_H)-1}\Big) && \text{by } \begin{cases} \text{[Seg24, Theorem 3.2 (ii)]} \\ \text{Theorem 4.1} \end{cases} \\
&= O^\sharp\Big(n^{\Omega(D_H)}\Big).
\end{aligned}$$

Comparing the two estimates for $\dim_k J_n$, we obtain $\dim_{\mathrm{Krull}} \gamma(H) - 1 = \Omega(D_H)$. □

5.3. The degree of $\gamma(H)$

In this subsection, we use the results of [Seg24, Section 4] to obtain more precise results concerning the strata $\gamma(H)$. We come back to where the proof of Theorem 1.2 ended, including all notation and hypotheses. Since we now know that $\dim_{\mathrm{Krull}} \gamma(H) - 1 = \Omega(D_H)$, we obtain (again by expressing $\dim_k J_n$ in two different ways):

$$\dim_k R_{n,H} = \mathrm{HF}_{R^H/(\Gamma_H \cap R^H)}(n) + O\Big(n^{\Omega(D_H)-1}\Big). \tag{5.3}$$

Assume that all non-factorizable elements of $\mathrm{Comp}_{\mathbb{P}^1(\mathbb{C})}(G, D, \xi)$ have the same degree W, and let N be their number. Then, $\operatorname{Proj} R$ embeds in the projective space $\mathbb{P}^{N-1}(k)$, and we can compute the degree[3] of the closed subset $\overline{\gamma(H)}$ (the degree of a non-irreducible subset is the sum of the degrees of its irreducible components of maximal Krull dimension). Note that we consider Proj instead of Spec: we have quotiented out by the action of $k^\times$ and all dimensions are one less than previously computed. In this case, we know that $\mathrm{HF}_{R^H/(\Gamma_H \cap R^H)}(n)$ and $\dim_k R_{n,H}$ vanish whenever n is not a multiple of W, and coincide with two polynomials of degree $\Omega(D_H)$ when evaluated at large enough multiples of W. Moreover, Equation (5.3) implies that the leading monomials of these two polynomials are equal:

$$\dim_k R_{Wn,H} \underset{n\to\infty}{\sim} \mathrm{HF}_{R^H/(\Gamma_H \cap R^H)}(Wn) \underset{n\to\infty}{\sim} \alpha \cdot (Wn)^{\Omega(D_H)} \quad \text{for some } \alpha > 0.$$

[3] A similar computation is possible when non-factorizable elements have different degrees – one should then average the coefficients in front of $n^{\Omega(D_H)}$ of the polynomials $Q_0, \dots, Q_{W-1}$ of Theorem 5.2 –, but there does not seem to be a standard notion of degree for subspaces of weighted projective spaces.

Classically, the number $\Omega(D_H)! \cdot \alpha$ is the degree of the closed subset $\overline{\gamma(H)}$ of $\mathbb{P}^{N-1}(k)$. Recall that the group $H_2(H, c_H)$ (mentioned in Subsection 4.1) is defined in [Woo21] or [Seg24, Definition 4.4]. The results of [Seg24] give the value of α, and consequently of the degree of $\overline{\gamma(H)}$, in various situations:

- If H is a non-splitter, then $\Omega(D_H) = 0$ by definition, and $\alpha = |H_2(H, c_H)|$ by [Seg24, Proposition 4.11]. In this case, $\overline{\gamma(H)}$ (embedded in the projective space $\mathbb{P}^{N-1}(k)$) is of dimension zero: it is a union of finitely many points. The degree is precisely the number of these points: there are $|H_2(H, c_H)|$ of them.

  Remark 5.4. If one sees $\overline{\gamma(H)}$ as embedded in affine space $\mathbb{A}^N(k)$ (as we have done in other sections), it is a union of $|H_2(H, c_H)|$ lines going through the origin 0.

- Assume that D consists of a single conjugacy class c of G and $\xi(c) = 1$. Let $\mathrm{ord}(c)$ be the order of any element contained in the conjugacy class c. Let s be the number of conjugacy classes of H contained in $c \cap H$, so that $\Omega(D_H) = s - 1$. By [Seg24, Corollary 4.14] and the comments underneath it, we have:
$$\alpha = \frac{\mathrm{ord}(c) \cdot |H_2(H, c_H)|}{(s-1)! \cdot |H^{\mathrm{ab}}|}.$$
  The degree of $\overline{\gamma(H)}$ is then given by:
$$\frac{\mathrm{ord}(c) \cdot |H_2(H, c_H)|}{|H^{\mathrm{ab}}|}.$$

6. Explicit description of the variety of components

This section is organized as follows:

- In Subsection 6.1, we describe a curve which is contained in each stratum $\gamma(H)$ (Theorem 6.2). In some situations, we prove that this curve is all of $\gamma(H)$.
- In Subsection 6.2, we define the notion of a free factor family of subgroups (Definition 6.4 and Definition 6.6). In Proposition 6.8, we state key properties of this notion.
- In Subsection 6.3, we describe the stratum corresponding to the product of a free factor family in terms of the strata associated to its factors (Proposition 6.14). We then use this to obtain a complete description of the variety of components under specific hypotheses (Theorem 6.16).
- In Subsection 6.4, we rephrase the description explicitly in terms of coordinates (Theorem 6.20).
- In Subsection 6.5, we apply Theorem 6.20 to obtain a complete description of the variety of components in the classical case of symmetric groups, using the results of [Seg23, Chapter 6].

6.1. A curve in each stratum

Fix a nontrivial D-generated subgroup H of G. In this subsection, we describe a curve which is contained in the stratum $\gamma(H)$, and give a criterion for it to be all of $\gamma(H)$. We start by defining ideals A_H° and A_H:

Definition 6.1. We let A_H° be the homogeneous ideal of R generated by the differences $m - m'$, for pairs (m, m') of components of same degree whose group is contained in H. We also define $A_H \coloneqq A_H^\circ + J_H^*$.

We let $p_1, \ldots, p_M$ be the non-factorizable components whose group is contained in H. In the quotient R/A_H, components whose group is not contained in H are made to vanish, and other components are made to be equal whenever their degrees match. In other words, whereas the ring of components R is the monoid algebra of the monoid of components $\mathrm{Comp}_{\mathbb{P}^1(\mathbb{C})}(G, D, \xi)$, the quotient R/A_H is the

monoid algebra of the submonoid of $\mathbb{Z}_{\geq 0}$ formed of integers which are the degree of some component whose group is contained in H. A more concrete description of the ideal A_H, of the quotient R/A_H, and of the closed subset $V(A_H) \subseteq \operatorname{Spec} R$ is given by the following theorem:

Theorem 6.2. *Let $p_1, \dots, p_M$ be the non-factorizable components whose group is contained in H, let $d(1), \dots, d(M)$ be their respective degrees, and let $D \coloneqq \gcd_{1 \leq i \leq M} d(i)$. Then:*

(i) The quotient R/A_H is isomorphic to the graded subalgebra $k[X^{d(1)}, \dots, X^{d(M)}]$ of $k[X]$.

(ii) A_H is a homogeneous prime ideal of R, and the Krull dimension of R/A_H is 1. Hence, $V(A_H)$ is a 1-dimensional irreducible closed subset of $\operatorname{Spec} R$, contained in $V(J_H^*)$.

(iii) We have $\sqrt{A_H + I_H} = \varpi$ and $V(A_H) \setminus \{\varpi\} \subseteq \gamma(H)$.

(iv) If there is at most one component of group H for each degree,[4] then $V(A_H) = \gamma(H) \cup \{\varpi\}$.

(v) Assume that k is algebraically closed and let $p_{M+1}, \dots, p_N$ be the non-factorizable components whose group is not contained in H. Embed the set of k-points of $\operatorname{Spec} R$ into k^N as in Paragraph 2.2.2. Then, the k-points of $V(A_H)$ form a "weighted" line, consisting of points of the following form, for some $\lambda \in k$:

$$(\underbrace{\lambda^{d(1)/D}, \lambda^{d(2)/D}, \dots, \lambda^{d(M)/D}}_{\substack{\text{non-factorizable components} \\ \text{of group } \subseteq H}}, \quad \underbrace{0, \dots, 0}_{\text{other non-factorizable components}}).$$

In particular, under the hypothesis of (iv), all the k-points of $\gamma(H)$ are points of the form above for $\lambda \in k^\times$.

Proof.

(i) The image of a component m (whose group is contained in H) in the quotient R/A_H is entirely determined by the degree of m, so that we can denote its image by $X^{\deg(m)}$. Applying this to each non-factorizable component whose group is contained in H shows that $R/A_H \simeq R^H/(A_H^\circ \cap R^H)$ is a quotient of $k[X^{d(1)}, \dots, X^{d(M)}]$. Relations between the non-factorizable components are generated by equalities of the form $m_1 \cdots m_u = m'_1 \cdots m'_u$ for non-factorizable components $m_1, \dots, m_u, m'_1, \dots, m'_u$ satisfying $\deg(m_1) + \cdots + \deg(m_u) = m'_1 + \cdots + \deg(m'_u)$. But such relations are automatically satisfied in $k[X^{d(1)}, \dots, X^{d(M)}]$, so that $R/A_H \simeq k[X^{d(1)}, \dots, X^{d(M)}]$.

(ii) The quotient $R/A_H \simeq k[X^{d(1)}, \dots, X^{d(M)}]$ is a graded subalgebra of the domain $k[X]$ and hence is integral. This proves that A_H is a prime ideal of R. Moreover, the Krull dimension of R/A_H equals the transcendence degree of its field of fractions $\operatorname{Frac}(R/A_H) \simeq \operatorname{Frac}(k[X^{d(1)}, \dots, X^{d(M)}]) = k(X^D)$, which is 1. The claims concerning $V(A_H)$ follow.

(iii) Since H is D-generated, the product $p_1 \cdots p_M$ has group exactly H and hence belongs to I_H. The image of that element in R/A_H is $X^{d(1)+\cdots+d(M)}$, and thus $R/(A_H + I_H)$ is a quotient of the algebra $k[X^{d(1)}, X^{d(2)}, \dots, X^{d(M)}]/(X^{d(1)+\cdots+d(M)})$, in which all homogeneous elements of positive degree are nilpotent. Therefore, the reduced quotient of $R/(A_H + I_H)$ is k, which implies that $\sqrt{A_H + I_H} = \varpi$. It follows that $V(A_H) \cap V(I_H) = \{\varpi\}$, i.e., that $V(A_H) \setminus \{\varpi\}$ is contained in $V(J_H^*) \setminus V(I_H) = \gamma(H)$.

(iv) We have shown that $V(A_H) \subseteq \gamma(H) \cup \{\varpi\}$, hence it remains only to show that $V(A_H) \supseteq \gamma(H)$, i.e., that $V(A_H) \cup V(I_H) = V(I_H^*)$. It suffices to show that $A_H \cap I_H = I_H^*$. Let S be the

[4] This hypothesis implies that H is a non-splitter and $H_2(H, c_H) = 1$ by [Seg24, Theorem 1.4]. This result is an "effective" version of the case $H_2(H, c_H) = 1$ of Remark 5.4.

submonoid of $\mathbb{Z}_{\geq 0}$ formed of integers which are the degree of some component of group H, and let F_k be the only component of group H of degree k for $k \in S$. An element of I_H is of the form:

$$x = \sum_{k \in S} \lambda_k F_k + \underbrace{x_{\supsetneq H}}_{\in I_H^*}.$$

The image of x in R/A_H (seen as a subring of $k[X]$ via point (i)) is $\sum_{k \in S} \lambda_k X^k$. So, x belongs to A_H exactly when the coefficients λ_k vanish for all $k \in S$, if and only if $x = x_{\supsetneq H}$ belongs to I_H^*.

(v) A point $(x_1, \ldots, x_N) \in (\operatorname{Spec} R)(k)$ is in $V(A_H)(k)$ if $x_{M+1} = \ldots = x_N = 0$ (because A_H contains J_H^*), and if two products of the coordinates x_i are equal whenever the sums of the degrees of the corresponding non-factorizable components p_i match. Using Bézout's identity, fix integers $a_1, \ldots, a_M \in \mathbb{Z}$ such that:

$$D = \sum_{i=1}^{M} a_i d(i).$$

For each $i \in \{1, \ldots, M\}$, we have in R/A_H:

$$p_i \left(\prod_{j \text{ s.t. } a_j \leq 0} p_j^{-a_j} \right)^{d(i)/D} = \left(\prod_{j \text{ s.t. } a_j \geq 0} p_j^{a_j} \right)^{d(i)/D}$$

because both sides are components of same degree, since $d(i) = \frac{d(i)}{D} \cdot \sum_j a_j d(j)$. Hence, $(x_1, \ldots, x_M, 0, \ldots, 0) \in V(A_H)(k)$ means that we have, for all $i \in \{1, \ldots, M\}$:

$$x_i = \left(\prod_j x_j^{a_j} \right)^{d(i)/D} = \lambda^{d(i)/D}$$

where $\lambda = \prod_j x_j^{a_j}$ does not depend on i. So, points of $V(A_H)(k)$ are of the form:

$$(\lambda^{d(1)/D}, \lambda^{d(2)/D}, \ldots, \lambda^{d(M)/D}, 0, \ldots, 0) \text{ with } \lambda \in k.$$

The coordinates of such a point satisfy the equalities defining R, since these are equalities between components of same degree and same group. □

Remark 6.3. The principle of Theorem 6.2 is to forget everything about a given component besides its degree and the fact that its group is contained in H. Finer descriptions of $\gamma(H)$ may be obtained by forgetting less information about these components, for example by replacing A_H by the ideal generated by the differences $m - m'$ between two components of same group H, same H-multidiscriminant (Definition 4.2) and same image in $H_2(H, c_H)$ (cf. Subsection 4.1), i.e., with the same lifting invariant in the sense of [Woo21].

6.2. Free families and factor families

Definition 6.4. A list $H_1, \ldots, H_k$ of subgroups of G is a *free family* if the two following conditions are met:

- For all disjoint subsets A, B of $\{1, \ldots, k\}$, we have $\langle (H_i)_{i \in A} \rangle \cap \langle (H_j)_{j \in B} \rangle = 1$.
- Elements of H_i commute with elements of H_j when $i \neq j$.

Definition 6.5. If $H_1, \ldots, H_k$ is a free family of subgroups of G, the (Zappa–Szép) *product* of this family is the subgroup $\langle H_1, \ldots, H_k \rangle$ of G.

As an abstract group, the product of a free family $H_1, \dots, H_k$ is isomorphic to the direct product of the groups H_i. We now fix a subgroup $H \in \mathrm{Sub}_{G,D}$.

Definition 6.6. A *factor family* of H is a list $H_1, \dots, H_k$ of D-generated subgroups of H such that every non-factorizable component whose group is contained in H has its group contained in some H_i.

Proposition 6.7. *Let $H_1, \dots, H_k$ be a factor family of H. Then, $H_1, \dots, H_k$ generate H.*

Proof. Use Proposition 2.11 to fix a component m of group H, and write m as a product of non-factorizable components $m = m_1 \cdots m_r$. Since $H_1, \dots, H_k$ is a factor family of H, each factor m_i has its group contained in some H_j. Hence, $H = \langle m \rangle = \langle \langle m_1 \rangle, \dots, \langle m_r \rangle \rangle$ is contained in $\langle H_1, \dots, H_k \rangle$. □

Proposition 6.8. *Let $H_1, \dots, H_k$ be a list of subgroups of H. Denote by Φ the following morphism of k-algebras:*

$$\Phi : \begin{cases} R^{H_1} \otimes \cdots \otimes R^{H_k} & \to & R^H \\ m_1 \otimes \cdots \otimes m_k & \mapsto & m_1 \cdots m_k \end{cases}.$$

Then:

(i) $H_1, \dots, H_k$ is a factor family if and only if Φ is surjective.

(ii) If $H_1, \dots, H_k$ is free, then Φ is injective.

(iii) Assume that $H_1, \dots, H_k$ is a free factor family of H. Then:

$$\Phi^{-1}(I_H \cap R^H) = (I_{H_1} \cap R^{H_1}) \otimes \cdots \otimes (I_{H_k} \cap R^{H_k}).$$

Proof.

(i) ($\Leftarrow$) Assume that Φ is surjective, and consider a non-factorizable component $m \in R^H$. Since m is in the image of Φ and is a component (not a linear combination thereof), it equals a product $m_1 \cdots m_k$ with $\langle m_i \rangle \subseteq H_i$. Since m is non-factorizable, we have $m = m_i$ for some i and thus $\langle m \rangle \subseteq H_i$.

($\Rightarrow$) Conversely, assume that $H_1, \dots, H_k$ is a factor family. To prove that Φ is surjective, it is enough to show that its image contains an arbitrary component m whose group is contained in H. Decompose m as a product $m_1 \cdots m_r$ of non-factorizable components. By hypothesis, for each $j \in \{1, \dots, r\}$, we can pick $\psi(j) \in \{1, \dots, k\}$ such that $\langle m_j \rangle \subseteq H_{\psi(j)}$. For $i \in \{1, \dots, k\}$, let m'_i be the product of the components m_j such that $\psi(j) = i$. Then $m = \Phi(m'_1 \otimes \cdots \otimes m'_k)$.

(ii) Assume that the family (H_i) is free. Since Φ maps pure tensors of components to components, and since components form a basis, it suffices to check that Φ maps distinct pure tensors of components to distinct components. Thus, we assume that there are two tuples $(m_1, \dots, m_k)$ and $(m'_1, \dots, m'_k)$ of components, with $\langle m_i \rangle$ and $\langle m'_i \rangle$ contained in H_i, such that:

$$\prod_{j=1}^{k} m_j = \prod_{j=1}^{k} m'_j.$$

Consider a braid relating two tuples representing these two components, and decompose it into elementary braids. Since the subgroups H_i commute with each other, elementary braids swapping two elements from different subgroups H_i do not change these elements, and do not affect the position of these elements among elements of the same subgroup. Hence, fixing some $i \in \{1, \dots, k\}$ and retaining only elementary braids which do exchange elements of the same subgroup H_i yields a braid relating m_i and m'_i, proving that $m_i = m'_i$. Therefore, Φ is injective.

(iii) First note that Φ is an isomorphism by the previous points. That $\Phi((I_{H_1} \cap R^{H_1}) \otimes \cdots \otimes (I_{H_k} \cap R^{H_k})) \subseteq I_H \cap R^H$ follows immediately from the fact that a product of components $m_1 \cdots m_k$ with $\langle m_i \rangle = H_i$ has group $\langle H_1, \dots, H_k \rangle = H$ (cf. Proposition 6.7).

Conversely, consider nonzero constants $\lambda_j \in k$ and components $m_{i,j}$ with $\langle m_{i,j} \rangle \subseteq H_i$ (such that no two tuples $(m_{i,j})_{1 \leq i \leq k}$ are equal for distinct values of j), such that:

$$\Phi\left(\sum_j \lambda_j \cdot (m_{1,j} \otimes \cdots \otimes m_{k,j})\right) \in I_H \cap R^H.$$

Then, each component $m_{1,j} \cdots m_{k,j}$ has group exactly H. Fix a j. The groups $\langle m_{i,j} \rangle$ for $i \in \{1, \dots, k\}$, being subgroups of H_i, form a free family of subgroups of H and hence the group generated by $m_{1,j} \cdots m_{k,j}$ is isomorphic to the direct product $\prod_{i=1}^k \langle m_{i,j} \rangle$. We have $|H| = \prod_{i=1}^k |H_i|$ but also $|H| = |\langle m_{1,j} \cdots m_{k,j} \rangle| = \prod_{i=1}^k |\langle m_{i,j} \rangle|$ with $|\langle m_{i,j} \rangle| \leq |H_i|$, which implies that the component $m_{i,j}$ has group exactly H_i, for each $i \in \{1, \dots, k\}$. Therefore, the element $\sum_j \lambda_j \cdot (m_{1,j} \otimes \cdots \otimes m_{k,j})$ belongs to $(I_{H_1} \cap R^{H_1}) \otimes \cdots \otimes (I_{H_k} \cap R^{H_k})$, which proves the claim. □

Finally, we give a group-theoretic criterion to identify some factor families:

Proposition 6.9. *Assume that $|D| = 1$ and $\xi = 1$, i.e., we consider a single conjugacy class c. Assume that H is a subgroup of G generated by subgroups $H_1, \dots, H_k$ such that:*

(i) For all $i \in \{1, \dots, k\}$, the subgroup H_i has a trivial intersection with $\langle (H_j)_{j \neq i} \rangle$.

(ii) $c \cap H = \bigsqcup_{i=1}^k (c \cap H_i)$.

Then, $H_1, \dots, H_k$ is a factor family of H.

Proof. For $i \in \{1, \dots, k\}$, let $c_i := c \cap H_i$. Consider a non-factorizable component m whose group is contained in H. By (ii), and using the fact that braids can permute conjugacy classes freely, we may choose a tuple $\underline{g}$ representing the component m of the form:

$$\underline{g} = (g_{1,1}, \dots, g_{1,n(1)}, \dots, g_{k,1}, \dots, g_{k,n(k)}) \qquad \text{with } g_{i,j} \in c_i.$$

Let $\pi_i := g_{i,1} \cdots g_{i,n(i)} \in H_i$. We have $\pi_1 \cdots \pi_k = 1$ and thus $\pi_i = (\pi_{i+1} \cdots \pi_k \pi_1 \cdots \pi_{i-1})^{-1} \in \langle (H_j)_{j \neq i} \rangle$. By (i), this implies $\pi_i = 1$. Therefore, all the tuples $\underline{g}_i = (g_{i,1}, \dots, g_{i,n(i)}) \in c_i^{n(i)}$ define components. Since m is non-factorizable, it equals one of its factors and thus $\langle m \rangle$ is contained in some H_i. □

6.3. The stratum associated to the product of a free factor family

We fix a subgroup $H \in \mathrm{Sub}_{G,D}$.

6.3.1. The set $\widetilde{\gamma}(H)$.

Definition 6.10. We denote by $\widetilde{\gamma}(H)$ the open subset of $\operatorname{Spec} R^H$ obtained as the complement of $V(I_H \cap R^H)$. In other words, $\widetilde{\gamma}(H)$ is the set of all prime ideals of $\operatorname{Spec} R^H$ for which some component m of group exactly H does not belong to p.

We denote by π_H the dominant map $\operatorname{Spec} R \to \operatorname{Spec} R^H$ induced by the inclusion $R^H \hookrightarrow R$, and by ι_H the embedding $\operatorname{Spec} R^H \to \operatorname{Spec} R$ induced by the surjection $R \twoheadrightarrow R/J_H^* \simeq R^H$. Since the composition $R^H \hookrightarrow R \twoheadrightarrow R/J_H^* \simeq R^H$ is the identity map, we have the equality:

$$\pi_H \circ \iota_H = \mathrm{id}_{\operatorname{Spec} R^H} \tag{6.1}$$

The sets $\gamma(H)$ and $\widetilde{\gamma}(H)$ determine each other, as shown below:

Proposition 6.11. *The sets $\gamma(H)$ and $\widetilde{\gamma}(H)$ are related by the following equalities:*

(i) $\gamma(H) = \iota_H(\widetilde{\gamma}(H))$

(ii) $\widetilde{\gamma}(H) = \pi_H(\gamma(H))$

(iii) $\gamma(H) = V(J_H^*) \cap \pi_H^{-1}(\widetilde{\gamma}(H))$

Proof.

(i) ($\subseteq$) Let $p \in \gamma(H)$. Then, p is a prime ideal of R containing J_H^*, and not containing some component m of group H. Since p contains J_H^*, we have $p = J_H^* + (p \cap R^H) = \iota_H(p \cap R^H)$. Moreover, $p \cap R^H$ is a prime ideal of R^H not containing m, and thus $p \cap R^H \in \widetilde{\gamma}(H)$.

($\supseteq$) Let $p \in \widetilde{\gamma}(H)$. Then, p is a prime ideal of R^H not containing some component m of group H. The prime ideal $\iota_H(p) = p + J_H^*$ of R contains J_H^* but does not contain m, and thus it belongs to $\gamma(H)$.

(ii) Follows from (i) by applying π_H to both sides, and using Equation (6.1).

(iii) The inclusion $\subseteq$ follows from Proposition 3.5 and from (ii). Conversely, let $p \in V(J_H^*) \cap \pi_H^{-1}(\widetilde{\gamma}(H))$. Since p contains J_H^*, we have $p = J_H^* + (p \cap R^H)$, i.e., $p = J_H^* + \pi_H(p)$. By hypothesis, $\pi_H(p) \in \widetilde{\gamma}(H)$ and hence there is some component m of group H which is not contained in $\pi_H(p) = p \cap R^H$. But then m is also not contained in p and thus $p \in \gamma(H)$. □

Remark 6.12. We can interpret Proposition 6.11 more concretely when k is algebraically closed and we focus on k-points. Let $p_1, \dots, p_M$ be the non-factorizable components whose group is contained in H. Then, Proposition 6.11 implies that the k-points of $\gamma(H)$ are exactly the points of the form

$$\Big(\underbrace{x_1, \dots, x_M}_{\substack{\text{coordinates corresponding to} \\ p_1, \dots, p_M}} , \underbrace{0, \dots, 0}_{\substack{\text{coordinates corresponding to} \\ \text{other non-factorizable components}}} \Big)$$

where $(x_1, \dots, x_M)$ is a k-point of $\widetilde{\gamma}(H)$. On the other hand, the k-points of $\pi_H^{-1}(\widetilde{\gamma}(H))$ are those whose coordinates $(x_1, \dots, x_M)$ form a k-point of $\widetilde{\gamma}(H)$, with no additional restriction on the other coordinates besides the fact that the point has to be a k-point of $\operatorname{Spec} R$.

6.3.2. Free factor families and strata. We now fix a free factor family $H_1, \dots, H_k$ of H. Our goal is to relate the stratum associated to H to the strata associated to its factors $H_1, \dots, H_k$. The first tool is the following proposition, which rephrases parts of Proposition 6.8 geometrically:

Proposition 6.13. *The natural morphism $\Phi\colon R^{H_1} \otimes \cdots \otimes R^{H_k} \to R^H$ is an isomorphism, i.e., the map $\operatorname{Spec} R^H \to \prod_{i=1}^k \operatorname{Spec} R^{H_i}$, which we also denote by Φ, is a homeomorphism (even an isomorphism of schemes). Moreover, we have the following equality:*

$$\widetilde{\gamma}(H) = \Phi^{-1}\left(\prod_{i=1}^k \widetilde{\gamma}(H_i) \right). \tag{6.2}$$

Proof. That Φ is an isomorphism follows from points (i) and (ii) of Proposition 6.8. Equation (6.2) follows directly from Proposition 6.8 (iii) by rephrasing it in terms of closed subsets and taking complements. □

Proposition 6.14. *We have the following description of $\gamma(H)$:*

$$\gamma(H) = V(J_H^*) \cap \bigcap_{i=1}^k \pi_{H_i}^{-1}(\widetilde{\gamma}(H_i)) = V(J_H^*) \cap \bigcap_{i=1}^k \pi_{H_i}^{-1}(\pi_{H_i}(\gamma(H_i))).$$

Proof. We compute $\gamma(H)$ in the following way:

$$\begin{aligned}\gamma(H) &= V(J_H^*) \cap \pi_H^{-1}(\widetilde{\gamma}(H)) && \text{by Proposition 6.11 (iii)}\\ &= V(J_H^*) \cap \pi_H^{-1}\left(\Phi^{-1}\left(\prod_{i=1}^{k} \widetilde{\gamma}(H_i)\right)\right) && \text{by Equation (6.2) .}\end{aligned}$$

Define $\pi_{H_i \to H}$ as the dominant map $\operatorname{Spec} R^H \to \operatorname{Spec} R^{H_i}$ induced by the inclusion $R^{H_i} \hookrightarrow R^H$, so that $\Phi = \pi_{H_1 \to H} \times \cdots \times \pi_{H_k \to H}$. The set $\prod_{i=1}^k \widetilde{\gamma}(H_i)$ can be tautologically described as the subset of $\prod_i \operatorname{Spec} R^{H_i}$ formed of elements whose projection in $\operatorname{Spec} R^{H_i}$ belongs to $\widetilde{\gamma}(H_i)$ for each $i \in \{1, \dots, k\}$. Therefore, the set $\Phi^{-1}\left(\prod_{i=1}^k \widetilde{\gamma}(H_i)\right)$ is the subset of $\operatorname{Spec} R^H$ formed of elements x whose projection in $\operatorname{Spec} R^{H_i}$ (namely $\pi_{H_i \to H}(x)$) belongs to $\widetilde{\gamma}(H_i)$ for all $i \in \{1, \dots, k\}$. This yields the equality:

$$\Phi^{-1}\left(\prod_{i=1}^{k} \widetilde{\gamma}(H_i)\right) = \bigcap_{i=1}^{k} \pi_{H_i \to H}^{-1}(\widetilde{\gamma}(H_i)).$$

Plugging this in the expression of $\gamma(H)$ obtained above, we get:

$$\begin{aligned}\gamma(H) &= V(J_H^*) \cap \pi_H^{-1}\left(\bigcap_{i=1}^{k} \pi_{H_i \to H}^{-1}(\widetilde{\gamma}(H_i))\right)\\ &= V(J_H^*) \cap \bigcap_{i=1}^{k} \pi_{H_i}^{-1}(\widetilde{\gamma}(H_i)) && \text{because } \pi_{H_i \to H} \circ \pi_H = \pi_{H_i}.\end{aligned}$$

The second equality follows using Proposition 6.11 (ii). □

6.3.3. Complete description of the variety of components. We make the following definition:

Definition 6.15. A subgroup $H \in \mathrm{Sub}_{G,D}$ is a *factored splitter* if there exists a free factor family of H of size at least 2.

Proposition 6.14 then implies the following theorem:

Theorem 6.16. *Assume that:*

- *Every nontrivial $H \in \mathrm{Sub}_{G,D}$ is either a non-splitter or a factored splitter.*
- *For every non-splitter $H \in \mathrm{Sub}_{G,D}$, there is at most one component of group H of each degree.*

Under these hypotheses, we describe $\gamma(H)$ for every $H \in \mathrm{Sub}_{G,D}$:

(i) $\gamma(1) = \{\varpi\}$.

(ii) If H is a non-splitter, $\gamma(H)$ is the curve $V(A_H) \setminus \{\varpi\}$ from Theorem 6.2.

(iii) Otherwise, H is a factored splitter, and we can write $H = H_1 \times \ldots \times H_k$ where the subgroups $H_1, \dots, H_k$ form a free factor family of non-splitters. Then:

$$\gamma(H) = V(J_H^*) \cap \bigcap_{i=1}^{k} \pi_{H_i}^{-1}\left(\pi_{H_i}(V(A_{H_i}) \setminus \{\varpi\})\right).$$

Since, by Theorem 1.1, the strata $\gamma(H)$ cover $\operatorname{Spec} R$, this yields a complete description of $\operatorname{Spec} R$.

Proof. Point (i) is clear. Consider a nontrivial subgroup $H \in \mathrm{Sub}_{G,D}$, and choose a maximal free factor family $H_1, \dots, H_k$ of H. For all $i \in \{1, \dots, k\}$, the subgroup H_i is a non-splitter, as otherwise H_i is a factored splitter (by the first hypothesis) and we can construct a longer free factor family, contradicting maximality. By Proposition 6.14, we can therefore reduce the proof of (iii) to the proof of (ii). Finally, point (ii) follows from Theorem 6.2 (iv) and from the second hypothesis. □

6.4. Complete description of the variety of components, in coordinates

In this subsection, we rephrase Theorem 6.16 in terms of coordinates. We let $p_1, \dots, p_N$ be the non-factorizable components and let $d(1), \dots, d(N)$ be their respective degrees.

We assume for the whole subsection that k is algebraically closed. As explained in Paragraph 2.2.2, we embed the set $(\operatorname{Spec} R)(k)$ of k-points of $\operatorname{Spec} R$ into the affine space k^N. We denote by $e_1, \dots, e_N$ the basis elements of k^N, each one corresponding to a non-factorizable component.

Definition 6.17. To each subgroup $H \in \mathrm{Sub}_{G,D}$, we associate the following point of k^N:

$$e_H \coloneqq \sum_{i \text{ such that } \langle p_i \rangle \subseteq H} e_i.$$

Moreover, we introduce the notion of "(strict) weighted span" of a set of points:

Definition 6.18. Let $\{x_1, \dots, x_n\}$ be a finite set of points of k^N, decomposed in the standard basis:

$$x_i = \sum_{j=1}^{N} \xi_{i,j} e_j.$$

The *weighted span* of $\{x_1, \dots, x_n\}$ is the set:

$$\left\{ \sum_{j=1}^{N} \sum_{i=1}^{n} \lambda_i^{d(j)} \xi_{i,j} e_j \;\middle|\; (\lambda_1, \dots, \lambda_n) \in k^n \right\}.$$

and the *strict weighted span* of $\{x_1, \dots, x_n\}$ is its weighted span, minus the weighted span of any proper subset, i.e.:

$$\left\{ \sum_{j=1}^{N} \sum_{i=1}^{n} \lambda_i^{d(j)} \xi_{i,j} e_j \;\middle|\; (\lambda_1, \dots, \lambda_n) \in (k^\times)^n \right\}.$$

Example 6.19. When the degrees $d(i)$ are all equal, the weighted span of $\{x_1, \dots, x_n\}$ is simply the linear subspace of k^N spanned by $x_1, \dots, x_n$. For a more interesting example, take $N = 2, d(1) = 1, d(2) = r$. The weighted span of the singleton $\{(1,1)\}$ is then the graph of $x \mapsto x^r$ in k^2.

We now use this terminology to rephrase Theorem 6.16 more explicitly:

Theorem 6.20. *Under the hypotheses of Theorem 6.16, the k-points of the strata $\gamma(H)$ admit the following description, for each $H \in \mathrm{Sub}_{G,D}$:*

- *The stratum $\gamma(1)$ contains a single k-point, namely the origin $(0, \dots, 0) \in k^N$.*
- *If H is nontrivial, then we can write H as the product of a free factor family $H_1, \dots, H_k$ of non-splitters. Then, the set of k-points of $\gamma(H) \subseteq \operatorname{Spec} R$ is, as a subset of k^N, the strict weighted span of the points $e_{H_1}, \dots, e_{H_k}$.*

Proof. The case $H = 1$ is clear. We consider the case of a nontrivial subgroup $H \in \mathrm{Sub}_{G,D}$. The hypotheses imply that H admits a free factor family $H_1, \dots, H_k$ of non-splitters. Let $p_{0,1}, \dots, p_{0,N(0)}$ be the non-factorizable components whose group is not contained in H and, for each $i \in \{1, \dots, k\}$, let $p_{i,1}, \dots, p_{i,N(i)}$ be the non-factorizable components whose group is contained in H_i. Since $H_1, \dots, H_k$ is a factor family, there are no other non-factorizable components, and since it is a free family these lists do not overlap. We may assume that the k-points of $\operatorname{Spec} R$, seen as points of k^N (where $N = N(0) + N(1) + \cdots + N(k)$), have their coordinates $x_{i,j}$ (corresponding respectively to the non-factorizable components $p_{i,j}$) ordered in the following way:

$$\Big(\underbrace{x_{0,1}, \dots, x_{0,N(0)}}_{\substack{\text{non-factorizable components} \\ \text{of group not contained in } H}} , \underbrace{x_{1,1}, \dots, x_{1,N(1)}}_{\substack{\text{non-factorizable components} \\ \text{of group contained in } H_1}} , \dots , \underbrace{x_{k,1}, \dots, x_{k,N(k)}}_{\substack{\text{non-factorizable components} \\ \text{of group contained in } H_k}} \Big).$$

Note that, for $i \in \{1, \dots, k\}$, the point e_{H_i} from Definition 6.17 has coordinates $x_{i',j} = \begin{cases} 1 & \text{if } i = i' \\ 0 & \text{otherwise.} \end{cases}$

The k-points of $\gamma(H)$ are, in particular, k-points of $V(J_H^*)$, and hence they satisfy $x_{0,1} = \dots = x_{0,N(0)} = 0$. By Proposition 6.14, the only additional condition that they satisfy is that, for each $i \in \{1, \dots, k\}$, their projection on $k^{N(i)}$ via the morphism π_{H_i}, which is the point $(x_{i,1}, \dots, x_{i,N(i)}) \in k^{N(i)}$, must be a k-point of $\widetilde{\gamma}(H_i)$.

Let $i \in \{1, \dots, k\}$. By hypothesis, there is at most one component of group H_i for each degree and each $i \in \{1, \dots, k\}$. Let $d(i,j)$ denote the degree of the non-factorizable component $p_{i,j}$. By Theorem 6.2 (v) and Proposition 6.11 (ii), the k-points of $\widetilde{\gamma}(H_i)$ are then exactly the points of $k^{N(i)}$ of the form $(\lambda_i^{d(i,1)}, \dots, \lambda_i^{d(i,N(i))})$ for some $\lambda_i \in k^\times$.

Putting everything together, the k-points of $\gamma(H)$ are the points of the form:

$$(\underbrace{0, \dots, 0}_{N(0)}, \underbrace{\lambda_1^{d(1,1)}, \dots, \lambda_1^{d(1,N(1))}}, \dots, \underbrace{\lambda_k^{d(k,1)}, \dots, \lambda_k^{d(k,N(k))}}) \quad \text{where } \lambda_1, \dots, \lambda_k \in k^\times.$$

The set of such points is exactly the strict weighted span of the points e_{H_i}, proving the claim. □

6.5. An application: the case of symmetric groups

We now give a concrete example where the variety of components can be described (and even drawn) using Theorem 6.20. In [Seg23, Chapter 6], we have focused on the following situation: G is the symmetric group $\mathfrak{S}_d$ for some $d \geq 2$, the set D is the singleton containing the conjugacy class c of transpositions, and ξ maps c to 1. A careful study of the action of braid groups on tuples of transpositions [Seg23, Theorem 6.2.6] shows that the following properties hold:

- The non-factorizable components of $\mathrm{Comp}_{\mathbb{P}^1(\mathbb{C})}(\mathfrak{S}_d, \{c\}, 1)$ are the orbits of tuples (τ, τ) where $\tau \in \mathfrak{S}_d$ is a transposition. In particular, there are $d(d-1)/2$ non-factorizable components, all of which have degree 2.
- The D-generated subgroups H of $\mathfrak{S}_d$ are all of the form $\mathfrak{S}_{A_1} \times \cdots \mathfrak{S}_{A_k}$ for some partition $A_1 \sqcup \dots \sqcup A_k$ of $\{1, \dots, d\}$. The factors $\mathfrak{S}_{A_i}$ are shown to form a free factor family of H using Proposition 6.9.
- If A is a subset of $\{1, \dots, d\}$, then the subgroup $\mathfrak{S}_A$ of $\mathfrak{S}_d$ is a non-splitter, and for each even degree $n \geq 2|A| - 2$ there is exactly one component of group $\mathfrak{S}_A$ and of degree n.

In particular, the hypotheses of Theorem 6.16/6.20 are satisfied, which leads to a description of the variety of components. We see it as embedded in the affine space $k^{\frac{d(d-1)}{2}}$, where we index the coordinates using pairs (i,j) with $1 \leq i < j \leq d$. If A is a subset of $\{1, \dots, d\}$, we define:

$$e_A := \sum_{\substack{1 \leq i < j \leq d \\ i,j \in A}} e_{i,j}$$

Then, Theorem 6.20 implies:

Theorem 6.21. *The subset* $(\operatorname{Spec} R)(k)$ *of* $k^{\frac{d(d-1)}{2}}$ *is the union of the vector spaces* $\operatorname{Span}_k(e_{A_1}, \dots, e_{A_k})$ *over (maximal) families* $\{A_1, \dots, A_k\}$ *of disjoint subsets of* $\{1, \dots, d\}$.

For the details, we refer to [Seg23, Section 6.4]. We now describe the set $V := (\operatorname{Proj} R)(k) \simeq (\operatorname{Spec} R)(k)/k^\times$ for small values of d:

($d = 3$) The set V embeds into $\mathbb{P}^2(k)$: it consists of four points, corresponding to (the lines spanned by) the points $e_{1,2}$, $e_{1,3}$, $e_{2,3}$ and $e_{1,2} + e_{1,3} + e_{2,3}$, i.e., to the subsets of $\{1,2,3\}$ of size ≥ 2.

($d = 4$) The set V embeds into $\mathbb{P}^5(k)$: it is the union of five points, corresponding to the points e_A for subsets $A \subseteq \{1, \dots, 4\}$ of size ≥ 3, and of two lines, corresponding to the two pairs of disjoint subsets of size 2.

($d = 5$) The set V embeds into $\mathbb{P}^9(k)$: it is the union of six points (for the subsets $A \subseteq \{1, \dots, 5\}$ of size ≥ 4), and of ten lines (for the pairs consisting of a subset of size 3 and its complement).

($d = 6$) The set V embeds into $\mathbb{P}^{14}(k)$: it is the (non-disjoint!) union of 22 points (for the subsets of $\{1, \dots, 6\}$ of size ≥ 4), of 10 lines (for the pairs of disjoint subsets of size 3) and of 15 planes (for the triples of disjoint subsets of size 2).

Data availability statement. No data are associated with this article.

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