DISCRIMINANTS AND LAST JUMPS

Let K be a local field (with finite residue field), K^{sep} be a separable closure of K, and $\Gamma_K := \mathrm{Gal}(K^{\mathrm{sep}}|K)$ be its absolute Galois group, with Γ_K^v ($v \geq -1$) the ramification filtration in the upper numbering. Consider a finite group G and a continuous homomorphism

$$\rho\colon \Gamma_K \to G$$

corresponding to an étale K-algebra L of dimension $|G|$. Let $H = \operatorname{im}\rho$. The K-algebra L is isomorphic to the product of $[G:H]$ copies of the field $L' := (K^{\mathrm{sep}})^{\ker\rho}$. The maximal order of L is the product of $[G:H]$ copies of $\mathcal{O}_{L'}$, so the relative discriminant of $L|K$ is the $[G:H]$-th power of that of $L'|K$.

Formula for the discriminant

The extension $L'|K$ is Galois of Galois group $\mathrm{Gal}(L'|K) = H$. By [Ser62, Chap. IV, §2, Prop. 4], we have the following expression of the valuation of the different of $L'|K$:

$$\begin{aligned}
v_{L'}(\mathfrak{D}_{L'|K}) &= \sum_{i=0}^{+\infty}\Big(|\mathrm{Gal}(L'|K)_i| - 1\Big) \\
&= \int_{-1}^{+\infty}\Big(|H_t| - 1\Big)\,\mathrm{d}t && \text{as } H_t = H_{\lceil t\rceil} = \mathrm{Gal}(L'|K)_t \\
&= \int_{-1}^{+\infty}\Big(|H^v| - 1\Big)[H^0 : H^v]\,\mathrm{d}v && \text{cf. [Ser62, Chap. IV, §3]} \\
&= \left|H^0\right|\int_{-1}^{+\infty}\left(1 - \frac{1}{|H^v|}\right)\mathrm{d}v \\
&= \left|\rho(\Gamma_K^0)\right|\int_{-1}^{+\infty}\left(1 - \frac{1}{|\rho(\Gamma_K^v)|}\right)\mathrm{d}v.
\end{aligned}$$

By [Ser62, Chap. III, §4, Prop. 6], we have for the discriminant

$$v_K(\mathfrak{d}_{L'|K}) = v_K(\mathrm{N}_{L'|K}(\mathfrak{D}_{L'|K})) = f_{L'|K}\cdot v_{L'}(\mathfrak{D}_{L'|K}).$$

Note that $[G:H]\cdot f_{L'|K}\cdot\left|\rho(\Gamma_K^0)\right| = [G:H]\cdot f_{L'|K}\cdot e_{L'|K} = [G:H]\cdot[L':K] = [G:H]|H| = |G|$. So

$$v_K(\mathfrak{d}_{L|K}) = [G:H]\cdot v_K(\mathfrak{d}_{L'|K}) = |G|\int_{-1}^{+\infty}\left(1 - \frac{1}{|\rho(\Gamma_K^v)|}\right)\mathrm{d}v.$$

Relation between discriminant and last jumps

Define the last jump in the ramification filtration, in both the upper and lower numbering:

$$\mathrm{lj}_{\mathrm{low}}(L|K) := \sup\left\{i \in \mathbb{Z}_{\geq -1} \;\middle|\; \mathrm{Gal}(L'|K)_i \neq 1\right\} \in \mathbb{Z}_{\geq -1}$$

$$\mathrm{lj}_{\mathrm{up}}(L|K) := \sup\left\{v \in \mathbb{R}_{\geq -1} \;\middle|\; \mathrm{Gal}(L'|K)^v \neq 1\right\} \in \frac{1}{|G|}\mathbb{Z}_{\geq -1}.$$

We have the following relation between these three invariants:

$$\begin{aligned}
v_K(\mathfrak{d}_{L|K}) &= |G|\int_{-1}^{+\infty}\left(1 - \frac{1}{|\rho(\Gamma_K^v)|}\right)\mathrm{d}v \\
&= |G|\int_{-1}^{\mathrm{lj}_{\mathrm{up}}(L|K)}\left(1 - \frac{1}{|\rho(\Gamma_K^v)|}\right)\mathrm{d}v \\
&= |G|\cdot(\mathrm{lj}_{\mathrm{up}}(L|K) + 1) - \int_{-1}^{\mathrm{lj}_{\mathrm{up}}(L|K)}[G : \rho(\Gamma_K^v)]\,\mathrm{d}v \\
&= |G|\cdot(\mathrm{lj}_{\mathrm{up}}(L|K) + 1) - [G : \rho(\Gamma_K^0)]\int_{-1}^{\mathrm{lj}_{\mathrm{up}}(L|K)}[\rho(\Gamma_K^0) : \rho(\Gamma_K^v)]\,\mathrm{d}v \\
&= |G|\cdot(\mathrm{lj}_{\mathrm{up}}(L|K) + 1) - [G : \rho(\Gamma_K^0)]\cdot(\mathrm{lj}_{\mathrm{low}}(L|K) + 1)
\end{aligned}$$

References

[Ser62] Jean-Pierre Serre. *Corps Locaux.* Hermann, Paris, 1962.