NO 3-FLIMSY SPACES: A SHORT PROOF

1. Proof of axiom (C3) (if anyone asks...)

Lemma 1.1. *Let X be a connected topological space, let $x \in X$, and let C be a connected component of $X \setminus \{x\}$. Then, $X \setminus C$ is connected.*

Proof. Let $X \setminus C \subseteq U \cup V$ with U, V open subsets of X and $U \cap V \subseteq C$. We have $X = C \cup U \cup V$. Without loss of generality, we have $x \in U$, and we want to show that $V \subseteq C$. By maximality of the connected component C, it is enough to show that $C \cup V$ is connected.

Assume that $C \cup V \subseteq M \cup N$ with M, N open subsets of X and $M \cap N \cap (C \cup V) = \emptyset$. Since C is connected and $C \subseteq M \cup N$, we have $C \subseteq M$ or $C \subseteq N$. Without loss of generality, $C \subseteq M$ (and then $N \cap C = \emptyset$), and our goal is to show that $N \cap V = \emptyset$. We have

$$X = C \cup U \cup V = (C \cup V) \cup U = M \cup N \cup U = M \cup U \cup N$$

but also

$$X = C \cup U \cup V = M \cup U \cup V$$

so $X = M \cup U \cup (N \cap V)$. By connectedness of X, since $M \cup U$ is open and non-empty, it suffices to show that $(M \cup U) \cap (N \cap V) = \emptyset$. First, $M \cap N \cap V = \emptyset$ by hypothesis, so it suffices to show that $U \cap N \cap V = \emptyset$. But $U \cap V \subseteq C$ so $U \cap N \cap V \subseteq N \cap C = \emptyset$. □

2. Connected subsets of 2-flimsy spaces

In this section, X is a 2-flimsy space.

Proposition 2.1. *Let $x, y \in X$ be distinct, and let C be a connected component of $X \setminus \{x, y\}$. Then, the three following sets are connected: $X \setminus \{x\} \setminus C$, $X \setminus \{y\} \setminus C$, $X \setminus C$.*

Proof. $X \setminus \{x\} \setminus C$ is connected by Lemma 1.1 applied in the connected space $X \setminus \{x\}$. The case of $X \setminus \{y\} \setminus C$ is symmetric. Note that $X \setminus \{x, y\} \setminus C \neq \emptyset$ (indeed, $X \setminus \{x, y\}$ is not connected so $\neq C$), so $X \setminus \{x\} \setminus C$ and $X \setminus \{y\} \setminus C$ intersect. Therefore, $X \setminus C = (X \setminus \{x\} \setminus C) \cup (X \setminus \{y\} \setminus C)$ is connected. □

Corollary 2.2. *Let $C \subsetneq X$ be a connected subset. Then, $X \setminus C$ is connected.*

Proof. Pick $y \in C$ and $z \in X \setminus C$. Using $C_{\ni y}(S)$ to mean "the connected component of y in S", we have

$$C = \bigcap_{x \in X \setminus C} C_{\ni y}(X \setminus \{x, z\})$$

Indeed:

($\subseteq$) C is connected, contains y, and is contained in $X \setminus \{x, z\}$ for all $x \in X \setminus C$

($\supseteq$) if $x \in X \setminus C$, then x does not belong to $C_{\ni y}(X \setminus \{x, z\})$ as it does not belong to $X \setminus \{x, z\}$

Hence:

$$X \setminus C = \bigcup_{x \in X \setminus C} (X \setminus C_{\ni y}(X \setminus \{x, z\}))$$

which is a union of connected subsets by Proposition 2.1, all of which contain z, so $X \setminus C$ is connected. □

Proposition 2.3. *Let $x, y \in X$ be distinct, and let C be a connected component of $X \setminus \{x, y\}$. Then:*

(i) $C \cup \{x\}$ and $C \cup \{y\}$ are connected

(ii) $C \cup \{x, y\}$ is connected

(iii) $X \setminus \{x, y\}$ has exactly two connected components

Proof. That $X \setminus \{x, y\}$ has exactly two connected components will follow if we show that the subset $(X \setminus \{x, y\}) \setminus C = X \setminus (C \cup \{x, y\})$ is connected, which by Corollary 2.2 reduces to (ii). Moreover, (ii) will follow from (i) since $C \cup \{x, y\} = (C \cup \{x\}) \cup (C \cup \{y\})$ and $C \neq \emptyset$. Since both cases of (i) are symmetric, we focus on proving that $C \cup \{x\}$ is connected. By Corollary 2.2, it suffices to show that $X \setminus (C \cup \{x\}) = X \setminus \{x\} \setminus C$ is connected, which is part of Proposition 2.1. □

Proposition 2.4. *Let $x, y \in X$ be distinct, and let $C \subseteq X \setminus \{x, y\}$ be a connected subset such that $C \cup \{x, y\}$ is connected. Then, C is a connected component of $X \setminus \{x, y\}$.*

Proof. By Corollary 2.2, the set $D := X \setminus (C \cup \{x, y\})$ is connected, so $X \setminus \{x, y\} = C \sqcup D$ where both C and D are connected. If C' is a connected subset of $X \setminus \{x, y\}$ properly containing C, then it meets D and $X \setminus \{x, y\} = C' \cup D$ would be connected, which it is not. We have shown that C is a maximal connected subset of $X \setminus \{x, y\}$, i.e., a connected component. □

3. There are no 3-flimsy spaces

Theorem 3.1. *There are no 3-flimsy spaces.*

Proof. Let X be a 3-flimsy space, let a, b, c be distinct points of X, and let C be a connected component of $X \setminus \{a, b, c\}$. By Proposition 2.1 applied in the 2-flimsy space $X \setminus \{a\}$, the subset $X \setminus \{a, b\} \setminus C$ is connected, and by Corollary 2.2 in $X \setminus \{b\}$, its complement $C \cup \{a\}$ is also connected. Similarly, $C \cup \{b\}$ and $C \cup \{c\}$ are connected, and $C \cup \{a, b\}$ is connected as a union. The set $C \cup \{a, b, c\}$ is not all of X because removing the three points a, b, c keeps it connected, so we may fix an $e \in X \setminus \{a, b, c\} \setminus C$.

C is a connected subset of $X \setminus \{e\} \setminus \{a, b\}$, and $C \cup \{a, b\}$ is connected, so by Proposition 2.4 the subset C is a connected component of $X \setminus \{a, b, e\}$. But $C \cup \{c\} \subseteq X \setminus \{a, b, e\}$ is connected, so by maximality of connected components we must have $c \in C$, which is not true. □