Project A4 – Combinatorial Euler products

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Context: Counting problems

We count certain algebraic/arithmetic objects x according to an invariant $\mathrm{invariant}(x) \in \mathbb{N}$:

$$|\{x \mid \mathrm{invariant}(x) \leq X\}| \underset{X \to \infty}{\sim} ???$$

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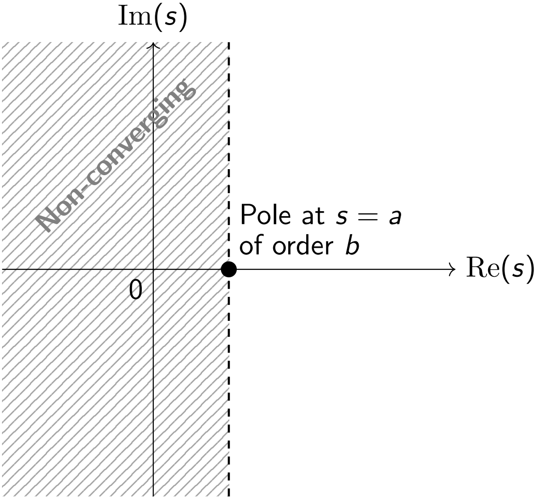
A general method: study the Dirichlet series

$$f(s) := \sum_{x} \mathrm{invariant}(x)^{-s}.$$

Tauberian theorems

Asymptotics of $N(X) := |\{x \mid \mathrm{invariant}(x) \leq X\}|$

$\leftrightarrow$ Rightmost pole of $f(s) = \sum \mathrm{invariant}(x)^{-s}$.



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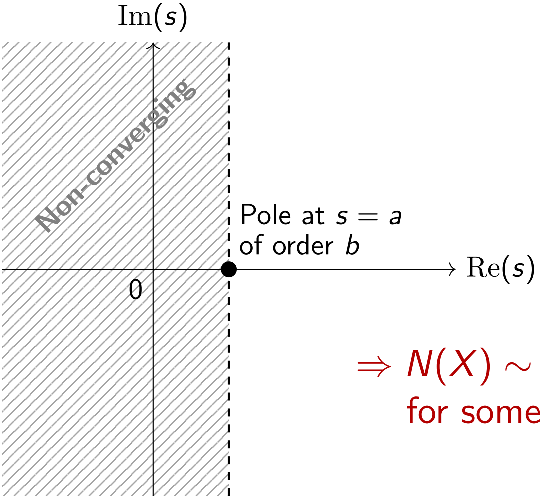

$\Rightarrow N(X) \sim CX^a(\log X)^{b-1}$
for some $C > 0$ ($\approx$ residue)

Local–global principles

Assume that we are counting objects over a number field K.

The field K has "places", i.e., completions. Example over $\mathbb{Q}$:

Places	**Archimedean**	**Primes (non-Archimedean)**					
	∞	2	3	5	7	11	...
Completions	$\mathbb{R}$	$\mathbb{Q}_2$	$\mathbb{Q}_3$	$\mathbb{Q}_5$	$\mathbb{Q}_7$	$\mathbb{Q}_{11}$	...

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A "global" object (over K) $\rightsquigarrow$ "local" objects (over completions).
Sometimes, this works backwards (**local–global principle**).
In this case, $f(s)$ factors as a **combinatorial Euler product**:

$$f(s) = \prod_{p \text{ place of } K} f_p(s).$$

where f_p counts local objects. We can then study the poles by comparison with classical Euler products, e.g., L-functions.

An example:

Count extensions of non-commutative fields/simple algebras.

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- There is a non-commutative version of Galois theory!
- There is a well-defined notion of discriminant!
- Made accessible by class field theory:
  Central simple algebras (=CSAs) over number fields are well-understood and satisfy a local–global principle.

A toy example (1/2)

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Example 1			X		X		
Example 2	X		X		X	X	

I had a third example, but...

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Example 3	🐘🐘🐘	X		X		X	

Unfortunately, the elephants are passing by and hiding a cell...
Can you see behind the elephants?

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Well done!

$\Rightarrow$ We can ignore both the place at ∞ and the parity condition.

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$\Rightarrow$ A quaternion algebra over $\mathbb{Q}$ is uniquely determined by its discriminant, which is the square of a squarefree integer.

(i.e., there are as many quaternion algebras over $\mathbb{Q}$ with discriminant $\leq X$ as there are squarefree integers $\leq \sqrt{X}$)

A toy example (2/2)

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The corresponding Dirichlet series is

$$f(s) := \sum_{n \text{ squarefree}} n^{-2s} = \prod_{p \text{ prime}} (1+p^{-2s}) = \prod_{p \text{ prime}} \frac{1-p^{-4s}}{1-p^{-2s}} = \frac{\zeta(2s)}{\zeta(4s)}$$

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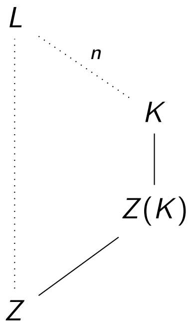
So the number of quaternion algebras with discriminant $\leq X$ is

$$\sim \frac{6}{\pi^2} X^{1/2}$$

More general extensions (1/2)

K a simple $\mathbb{Q}$-algebra, $Z \subseteq Z(K)$, $n \geq 2$.
We count extensions $L|K$ of degree n with center Z.



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$\Leftrightarrow$ Describe poles (location, order) of Dirichlet series

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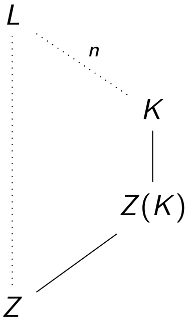

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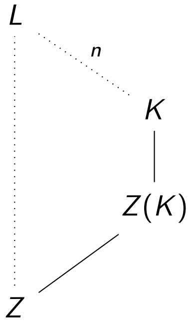

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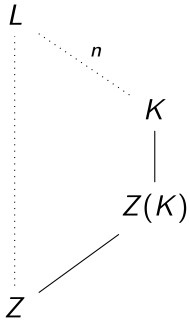

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Comparing this Euler product with a zeta function, we prove our main theorem...

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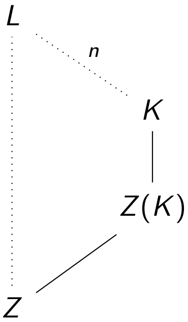
Theorem (Gundlach–S. '24)

For explicit $a, b \in \mathbb{Q}_{>0}$, $C \in \mathbb{R}_{\geq 0}$, the number of extensions $L|K$ with $Z(L) = Z$, $[L : K] = n$, and $\|\mathrm{Disc}(L)\| \leq X$ is $\underset{X\to\infty}{\sim} CX^a(\log X)^{b-1}$.



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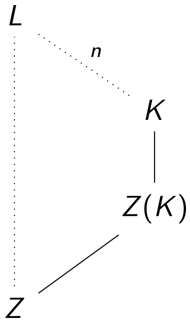
Assume $Z(K)|Z$ is Galois of group G.
Let $M := \sqrt{n[K : Z]}$, $j := \sqrt{\frac{n}{[Z(K):Z]}}$.
(They have to be integers for L to exist.)
$u :=$ smallest prime divisor of $j \cdot |G|$.
$\beta :=$ proportion of $g \in G$ with $u \mid j \cdot \mathrm{ord}(g)$

$$a = M^2\left(1 - \frac{1}{u}\right) \qquad b = (u-1)\beta.$$

The expression for C is more complicated.



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Other work within Project A4

Combinatorial Euler products are also used to count:

- representations of arithmetic and profinite groups (Blomer, Voll)
- average kernel sizes of module representations over finite Artinian rings (Rossmann, Voll)
- wildly ramified extensions of function fields in characteristic p (Gundlach, Potthast, S.)

Come check our poster!