THE RELATIVE LAST JUMP FORMULA

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1. The relative last jump formula

Definition 1.1. If $L|K$ is a Galois extension of local fields, we define

$$\mathrm{lastjump}_{\mathrm{low}}(L|K) = \sup\{i \in \mathbb{N}_{\geq -1} \mid \mathrm{Gal}(L|K)_i \neq 1\} \in \mathbb{N}_{\geq -1} \cup \{+\infty\}$$
$$\mathrm{lastjump}_{\mathrm{up}}(L|K) = \sup\{v \in \mathbb{R}_{\geq -1} \mid \mathrm{Gal}(L|K)^v \neq 1\} \in \tfrac{1}{|\mathrm{Gal}(L|K)|}\mathbb{N}_{\geq -1} \cup \{+\infty\}$$

(See [Ser62] for more information about the ramification filtration, in both lower and upper numbering.)

By definition, we have

$$\mathrm{lastjump}_{\mathrm{low}}(L|K) = \psi_{L|K}\big(\mathrm{lastjump}_{\mathrm{up}}(L|K)\big) \qquad \text{and} \qquad \mathrm{lastjump}_{\mathrm{up}}(L|K) = \varphi_{L|K}\big(\mathrm{lastjump}_{\mathrm{low}}(L|K)\big)$$

where $\psi_{L|K}$, $\varphi_{L|K}$ denote the (inverse) Herbrand functions. Moreover:

- $\mathrm{lastjump}_{\mathrm{low}}(L|K) = -1 \iff \mathrm{lastjump}_{\mathrm{up}}(L|K) = -1 \iff L|K$ is unramified,
- $\mathrm{lastjump}_{\mathrm{low}}(L|K) = 0 \iff \mathrm{lastjump}_{\mathrm{up}}(L|K) = 0 \iff L|K$ is tamely ramified.

Let $K^{(v)}$ be the fixed subfield of K^{sep} under $\mathrm{Gal}(K^{\mathrm{sep}}|K)^v$. Then, we have $\mathrm{lastjump}_{\mathrm{up}}(L|K) < v \iff L \subseteq K^{(v)}$. For example, the maximal tamely ramified extension of K is $\bigcap_{v>0} K^{(v)}$.

If L, L' are two Galois extensions of K, then the last jump of their compositum can be computed by

$$\mathrm{lastjump}_{\mathrm{up}}(LL'|K) = \max\big(\mathrm{lastjump}_{\mathrm{up}}(L|K),\ \mathrm{lastjump}_{\mathrm{up}}(L'|K)\big).$$

We now prove a relative last jump formula analogous to the relative discriminant formula:

Lemma 1.2 (Relative last jump formula). *Let $L|E|K$ be a tower of separable extensions. We have*

$$\mathrm{lastjump}_{\mathrm{up}}(L|K) = \max\big(\mathrm{lastjump}_{\mathrm{up}}(E|K),\ \varphi_{E|K}(\mathrm{lastjump}_{\mathrm{up}}(L|E))\big)$$
$$\mathrm{lastjump}_{\mathrm{low}}(L|K) = \max\big(\psi_{L|E}(\mathrm{lastjump}_{\mathrm{low}}(E|K)),\ \mathrm{lastjump}_{\mathrm{low}}(L|E).\big)$$

Proof. The second formula follows from the first by applying the (increasing) function $\psi_{L|K} = \psi_{L|E} \circ \psi_{E|K}$, so we focus on the first formula. Consider the short exact sequence

$$1 \to \mathrm{Gal}(L|E) \to \mathrm{Gal}(L|K) \xrightarrow{\pi} \mathrm{Gal}(E|K) \to 1.$$

For any $v \geq -1$, we have $\pi(\mathrm{Gal}(L|K)^v) = \mathrm{Gal}(E|K)^v$ by [Ser62, Chap. IV, §3, Prop. 14]. In particular, if $v \leq \mathrm{lastjump}_{\mathrm{up}}(E|K)$ then $\mathrm{Gal}(L|K)^v \neq 1$, so $\mathrm{lastjump}_{\mathrm{up}}(L|K) \geq \mathrm{lastjump}_{\mathrm{up}}(E|K)$. Now, assume that $v > \mathrm{lastjump}_{\mathrm{up}}(E|K)$, so that $\pi(\mathrm{Gal}(L|K)^v) = \mathrm{Gal}(E|K)^v = 1$, i.e., $\mathrm{Gal}(L|K)^v \subseteq \mathrm{Gal}(L|E)$. Then:

$$\begin{aligned}
\mathrm{Gal}(L|K)^v &= \mathrm{Gal}(L|K)^v \cap \mathrm{Gal}(L|E) \\
&= \mathrm{Gal}(L|K)_{\psi_{L|K}(v)} \cap \mathrm{Gal}(L|E) \\
&= \mathrm{Gal}(L|E)_{\psi_{L|K}(v)} && \text{by [Ser62, Chap. IV, §1, Prop. 3 et Cor.]} \\
&= \mathrm{Gal}(L|E)^{\varphi_{L|E}(\psi_{L|K}(v))} \\
&= \mathrm{Gal}(L|E)^{(\varphi_{L|E}\circ\psi_{L|E}\circ\psi_{E|K})(v)} && \text{by [Ser62, Chap. IV, §3, Prop. 15]} \\
&= \mathrm{Gal}(L|E)^{\psi_{E|K}(v)}
\end{aligned}$$

Date: July 29, 2025.

so, under the hypothesis $v > \mathrm{lastjump}_{\mathrm{up}}(E|K)$, we have $\mathrm{Gal}(L|K)^v = 1$ if and only if $\psi_{E|K}(v) > \mathrm{lastjump}_{\mathrm{up}}(L|E)$, if and only if $v > \varphi_{E|K}(\mathrm{lastjump}_{\mathrm{up}}(L|E))$. □

2. A criterion for relative tameness

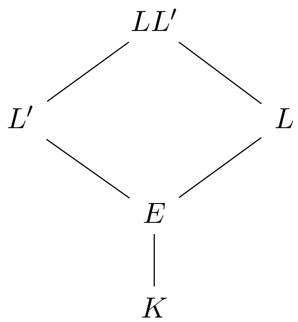
Definition 2.1. Consider a tower of Galois field extensions of local fields $L|E|K$, with $L|K$ Galois. We say that $L|E$ *does not make* $E|K$ *wilder* if $\mathrm{lastjump}_{\mathrm{up}}(L|K) = \mathrm{lastjump}_{\mathrm{up}}(E|K)$.

An immediate consequence of the relative last jump formula is:

Corollary 2.2. *The extension $L|E$ does not make $E|K$ wilder if and only if*

$$\mathrm{lastjump}_{\mathrm{up}}(L|E) \leq \mathrm{lastjump}_{\mathrm{low}}(E|K).$$

Lemma 2.3. *Consider a diagram of extensions, all Galois over K:*



If $L|E$ does not make $E|K$ wilder, then $LL'|L'$ does not make $L'|K$ wilder.

Proof. Let v be the last jump of $L'|K$. For any $v' > v$, we have $E \subseteq L' \subseteq K^{(v')}$ and, since $L|E$ does not make $E|K$ wilder, we have $L \subseteq K^{(v')}$, so the compositum LL' is contained in $K^{(v')}$. □

References

[Ser62] Jean-Pierre Serre. *Corps Locaux.* Hermann, Paris, 1962.