Asymptotics of wildly ramified extensions of function fields

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All results joint with Fabian Gundlach

Journées Arithmétiques, Belval, Luxembourg
June 30th, 2025

Background: counting field extensions

Question

K a field, G a finite group. Asymptotics as $X \to \infty$ of

$$\left| \left\{ L|K \text{ Galois} \;\middle|\; \mathrm{Gal}(L|K) \simeq G, \quad \|\mathrm{Disc}(L|K)\| \leq X \right\} \right|$$

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Now, $q = p^d$ is a prime power, $K = \mathbb{F}_q(T)$ a **rational function field**.

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If G is a p-group (wild extensions): abelian case $\approx$ solved, general case mysterious...

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We assume:

- G has nilpotency class ≤ 2, i.e., $G/Z(G)$ is abelian

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Theorem (Lazard correspondence)

There exists a Lie $\mathbb{F}_p$-algebra $\mathfrak{g}$ such that

$$G \simeq (\mathfrak{g}, \circ) \qquad \text{where } \circ \text{ is the group law } x \circ y := x + y + \frac{1}{2}[x, y]$$

$\rightsquigarrow$ the Lie algebra $\mathfrak{g}$ can "play the role" of the group G

Example of the Lazard correspondence

Example: the Heisenberg group $G = H(\mathbb{F}_p) = \begin{pmatrix} 1 & \mathbb{F}_p & \mathbb{F}_p \\ & 1 & \mathbb{F}_p \\ & & 1 \end{pmatrix}$

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In that case, when we turn it back into a group, we get:

$$(\mathfrak{g} \otimes K, \circ) \simeq \begin{pmatrix} 1 & K & K \\ & 1 & K \\ & & 1 \end{pmatrix} = H(K)$$

$\rightsquigarrow$ simply the Heisenberg group over K!

Parametrization of G-extensions

K a field of characteristic p, $\overline{K}$ a separable closure, $\Gamma_K = \mathrm{Gal}(\overline{K}|K)$.

Advantage of Lie algebras: we can **base-change** the group $G = (\mathfrak{g}, \circ)$:

$$G_K := (\mathfrak{g} \otimes_{\mathbb{F}_p} K, \circ) \qquad G_{\overline{K}} := (\mathfrak{g} \otimes_{\mathbb{F}_p} \overline{K}, \circ)$$

These are equipped with a Frobenius σ (induced by $x \mapsto x^p$). Fixed points = G.

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$$\underbrace{H^1(\Gamma_K, G)}_{G\text{-extensions of } K} \simeq \underbrace{G_K /\!/ _{G_K}}_{\text{set of orbits}} \quad \text{for the } \sigma\text{-conjugation action } g.m = \sigma(g)mg^{-1}$$

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By what invariant are we going to count? $\rightsquigarrow$ we need to control ramification

The ramification filtration and the last jump

$$
\begin{array}{ccccc}
 & & G = \mathrm{Gal}(L|K) & & \\
 & \nearrow & \uparrow & \nwarrow & \\
\mathcal{D}_{\mathfrak{p}} & & \mathcal{D}_{\mathfrak{q}} & & \cdots \\
\trianglerighteq & & \trianglerighteq & & \trianglerighteq \\
\mathcal{I}_{\mathfrak{p}} & & \mathcal{I}_{\mathfrak{q}} & & \vdots \\
\supsetneq & & \supsetneq & & \\
\mathcal{I}_{\mathfrak{p}}^{\left(1^{\mathrm{st}}\mathrm{jump}_{\mathfrak{p}}\right)} & & \mathcal{I}_{\mathfrak{q}}^{\left(1^{\mathrm{st}}\mathrm{jump}_{\mathfrak{q}}\right)} & & \\
\supsetneq & & \supsetneq & & \\
\mathcal{I}_{\mathfrak{p}}^{\left(2^{\mathrm{nd}}\mathrm{jump}_{\mathfrak{p}}\right)} & & \vdots & & \\
\supsetneq & & \supsetneq & & \\
\vdots & & \mathcal{I}_{\mathfrak{q}}^{\left(\mathrm{lastjump}_{\mathfrak{q}}(L|K)\right)} & & \\
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$\rightsquigarrow$ decomposition groups at various primes of K

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$\rightsquigarrow$ higher inertia

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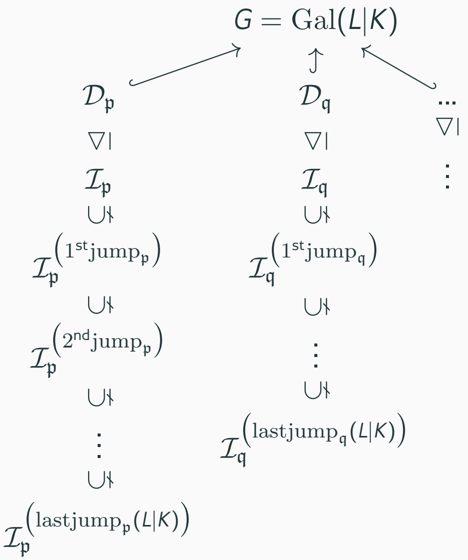

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$$G = \mathrm{Gal}(L|K)$$

$$\mathcal{D}_\mathfrak{p} \hookrightarrow G, \quad \mathcal{D}_\mathfrak{q} \hookrightarrow G, \quad \dots \hookrightarrow G$$

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We define the global invariant:

$$\mathrm{lastjump}(L|K) = \sum_{\mathfrak{p}} \deg \mathfrak{p} \cdot \mathrm{lastjump}_\mathfrak{p}(L|K)$$

(when G is abelian, $\approx$ the conductor)

Main results

G = finite p-group of nilpotency class ≤ 2, $K = \mathbb{F}_q(T)$ for $q = p^d$

Goal

Count G-extensions $L|K$ with $\mathrm{lastjump}(L|K) = X$.

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Our main result is a local–global principle, phrased in terms of generating functions:

Theorem (Gundlach, S., 2025)

$$\sum_{\substack{L|K \\ G\text{-extension}}} \frac{X^{\mathrm{lastjump}(L|K)}}{|\mathrm{Aut}(L|K)|} = \prod_{\mathfrak{p} \text{ prime}} \sum_{\substack{L_\mathfrak{p}|K_\mathfrak{p} \\ G\text{-extension}}} \frac{X^{\deg \mathfrak{p} \cdot \mathrm{lastjump}_\mathfrak{p}(L|K)}}{|\mathrm{Aut}(L_\mathfrak{p}|K_\mathfrak{p})|}$$

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Consequence: local counting $\Rightarrow$ global counting

Local counting 1/2

Let $K = \mathbb{F}_q((T))$ (*local* function field), $v > 0$ a positive real.

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From Abrashkin's *nilpotent Artin–Schreier theory*, we deduce:

Count G-extensions with $\mathrm{lastjump}(L|K) < v$

$$\iff$$

Count solutions of certain equations with indeterminates in the Lie $\mathbb{F}_q$-algebra $\mathfrak{g} \otimes \mathbb{F}_q$

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In terms of coordinates: polynomial equations over $\mathbb{F}_q$ (depending on $\mathfrak{g}$)

$\rightsquigarrow$ count the $\mathbb{F}_q$-points of an algebraic variety over $\mathbb{F}_p$ (depending on $\mathfrak{g}$ and v)

Local counting 2/2

Theorem (Gundlach, S. 2025)

For some of these groups G, when $K = \mathbb{F}_q(T)$, we prove estimates of the form

$$\sum_{\substack{L|K \ G\text{-extension} \\ \mathrm{lastjump}(L|K)=N}} \frac{1}{|\mathrm{Aut}(L|K)|} = C(N) \cdot q^{AN} \cdot N^{B-1} + o(q^{AN} \cdot N^{B-1}) \qquad \text{as } N \to +\infty$$

C is a periodic function $\mathbb{Q} \to \mathbb{R}_{\geq 0}$ with $C(0) > 0$, and $A, B \in \mathbb{Q}$ are explicit.

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- we consider more general Heisenberg groups... (for $p > 3$, we always have $B = 1$)

Thanks for your attention!

arXiv: 2502.18207

EXTRA SLIDE: the equations

Count G-ext. with $\mathrm{lastjump}(L|K) < v \iff$ Count solutions of the following equations...

Variables: $D_b \in \mathfrak{g} \otimes \mathbb{F}_q$ for $p \nmid b < v$

Equations:

- For any $p \nmid b < v$, $\sum_{\substack{p \nmid b_1, b_2 < b \\ b_1 + b_2 = b}} b_1 [D_{b_1}, D_{b_2}] = 0$
- For any $i \geq 1$ and any $p \nmid b \geq vp^i$, $\sum_{\substack{p \nmid b_1, b_2 < v \\ b_1 p^i + b_2 = b}} b_1 [\sigma^i(D_{b_1}), D_{b_2}] = 0$