Lifting morphisms with the power of choice

September 28, 2019

Let $p : A \to B$ be a morphism of rings and $I = \ker(p)$. We assume that p is surjective and that $I^2 = 0$.

Let $\rho : G \to GL_n(B)$ be a group morphism.

1 An action of G on $\mathfrak{M}_n(I)$

We define an action of G on $\mathfrak{M}_n(I)$ in the following way : if $\rho' : G \to GL_n(A)$ is a *map* (nothing more !) lifting $GL_n(B)$, we write :

$$g.M = \rho'(g)M\rho'(g)^{-1}$$

which indeed is in $\mathfrak{M}_n(I)$. Let's show that it doesn't depend on ρ'. Let ρ'_2 be another set-theoretical lift of ρ. Then :

$$p_*(\rho'(\rho'_2)^{-1}) = \rho\rho^{-1} = I_n$$

so $\rho'(g)(\rho'_2)^{-1}(g) = I_n + M(g)$ with $M(g) \in \mathfrak{M}_n(I)$. We have

$$\rho'(g)^{-1} = \rho'_2(g)^{-1}[I_n + M(g)]^{-1}$$

and since $[I_n + M(g)][I_n - M(g)] = I_n - M(g)^2 = I_n$ we can write

$$\rho'(g)^{-1} = \rho'_2(g)^{-1}[I_n - M(g)]$$

So :

$$\rho'(g)M\rho'(g)^{-1} = [I_n+M(g)]\rho'_2(g)M\rho'_2(g)^{-1}[I_n-M(g)] = \rho'_2(g)M\rho'_2(g)^{-1}+(\text{stuff in } I^2)$$

Now, a very classic argument shows that since $g.M$ doesn't depend on ρ', the axiom of choice (which seems required to show that such a ρ' exists) is not needed to define this action.

When $\mathfrak{M}_n(I)$ is equipped by this action, we shall call it $\mathrm{ad}(\rho)$.

2 Measuring non-homomorphicity with 2-cocycles

Assume $\rho' : G \to GL_n(A)$ is a *map* (nothing more !) lifting $GL_n(B)$, that is such that $p_*(\rho') = \rho$.

Then we can associate the following to ρ' :

$$d(a,b) = \rho'(ab)\rho'(b)^{-1}\rho'(a)^{-1}$$

This is a map $G^2 \to GL_n(A)$, and moreover :

$$p(d(a,b)) = \rho(ab)\rho(b)^{-1}\rho(a)^{-1} = I_n$$

and so $d(a,b)$ is of the form $I_n + e(a,b)$ with $e : G^2 \to \mathfrak{M}_n(I)$.
We can write :

$$\rho'(ab) = d(a,b)\rho'(a)\rho'(b) = (I_n + e(a,b))\rho'(a)\rho'(b)$$

Now we can compute $\rho'(abc)$ in two different ways :

$$\begin{aligned}\rho'((ab)c) &= d(ab,c)\rho'(ab)\rho'(c)\\ &= d(ab,c)d(a,b)\rho'(a)\rho'(b)\rho'(c)\\ &= [I_n + e(ab,c)][I_n + e(a,b)]\rho'(a)\rho'(b)\rho'(c)\end{aligned}$$

$$\begin{aligned}\rho'(a(bc)) &= d(a,bc)\rho'(a)\rho'(bc)\\ &= d(a,bc)\rho'(a)d(b,c)\rho'(b)\rho'(c)\\ &= d(a,bc)\rho'(a)d(b,c)\rho'(a)^{-1}\rho'(a)\rho'(b)\rho'(c)\\ &= [I_n + e(a,bc)][I_n + \rho'(a)e(b,c)\rho'(a)^{-1}]\rho'(a)\rho'(b)\rho'(c)\end{aligned}$$

Now these have to be equal, and ρ' goes into $GL_n(A)$ so :

$$[I_n + e(ab,c)][I_n + e(a,b)] = [I_n + e(a,bc)][I_n + \rho'(a)e(b,c)\rho'(a)^{-1}]$$
$$I_n + e(ab,c) + e(a,b) + e(ab,c)e(a,b) = I_n + e(a,bc) + \rho'(a)e(b,c)\rho'(a)^{-1} + e(a,bc)\rho'(a)e(b,c)\rho'(a)^{-1}$$
$$e(ab,c) + e(a,b) = e(a,bc) + \rho'(a)e(b,c)\rho'(a)^{-1}$$

(We used that $I^2 = 0$ to cancel the $e \times e$ terms)
This amounts to saying that e is a 2-cocycle for the action of G on $\mathrm{ad}(\rho)$.
Now if ρ'_2 is another lifting of ρ (and e_2 the associated 2-cocycle), let $m = \rho'_2 - \rho'$. We have $m : G \to \mathfrak{M}_n(I)$, and :

$$
\begin{aligned}
e_2(g_1,g_2) =& \rho_2'(g_1g_2)\rho_2'(g_2)^{-1}\rho_2'(g_1)^{-1} - I_n \\
=& \rho'(g_1g_2)\rho'(g_2)^{-1}\rho'(g_1)^{-1} - I_n \\
& + m(g_1g_2)\rho'(g_2)^{-1}\rho'(g_1)^{-1} \\
& + \rho'(g_1g_2)\rho'(g_2)^{-1}m(g_2)\rho'(g_2)^{-1}\rho'(g_1)^{-1} \\
& + \rho'(g_1g_2)\rho'(g_2)^{-1}\rho'(g_1)^{-1}m(g_1)\rho'(g_1)^{-1} \\
& + (\text{stuff in } I^2) \\
=& e(g_1,g_2) \\
& + m(g_1g_2)\rho'(g_2g_1)^{-1} + (\text{stuff in } I^2) \\
& + \rho'(g_1)\rho'(g_2)\rho'(g_2)^{-1}m(g_2)\rho'(g_2)^{-1}\rho'(g_1)^{-1} + (\text{stuff in } I^2) \\
& + \rho'(g_1)\rho'(g_2)\rho'(g_2)^{-1}\rho'(g_1)^{-1}m(g_1)\rho'(g_1)^{-1} + (\text{stuff in } I^2) \\
=& e(g_1,g_2) \\
& + m(g_1g_2)\rho'(g_2g_1)^{-1} \\
& + \rho'(g_1)m(g_2)\rho'(g_2)^{-1}\rho'(g_1)^{-1} \\
& + m(g_1)\rho'(g_1)^{-1}
\end{aligned}
$$

So if we define $\alpha(g) = m(g)\rho'(g)^{-1}$ we have :

$$
e_2(g_1,g_2) - e(g_1,g_2) = \alpha(g_1g_2) + g_1.\alpha(g_2) + \alpha(g_1)
$$

Which is exactly to say that $e_2 - e$ is a 2-coboundary. The same computations read backwards show that every 2-coboundary defines similarly another lifting of ρ.

3 What happens if I assume choice ?

When we assume choice, the surjection $GL_n(A) \to GL_n(B)$ admits a section, and so there always exists at least one set-theoretical lifting of ρ. In that case, the (non-empty) set of all set-theoretic liftings of ρ is exactly one cohomology class in :

$$
H^2(G, \mathrm{ad}(\rho))
$$

If this cohomology class is trivial (e.g. when $H^2(G, \mathrm{ad}(\rho)) = 0$), this means that the 2-cocycle 0 is in it, so there is a set-theoretic lifting ρ' of ρ such that the corresponding e is zero, that is to say ρ' is a group morphism.

We have a theorem :

Theorem 1 (in ZFC)**.** *Let $p : A \to B$ be a morphism of rings and $I = \ker(p)$. We assume that p is surjective and that $I^2 = 0$. Let $\rho : G \to GL_n(B)$ be a group morphism, and assume :*

$$
H^2(G, \mathrm{ad}(\rho)) = 0
$$

Then there exists a group morphism $\rho' : G \to GL_n(A)$ such that $\rho = p_*(\rho')$.

4 Is choice required ?

When we look at what happens before, it seems that choice is only useful to show that the cohomology class we're speaking of is well-defined. This inspires the following question :

Question 1. *Are there models of ZF in which theorem 1 doesn't hold ?*

The fact that the definition of $\mathrm{ad}(\rho)$ seems to require choice but in fact doesn't might be a clue that this is not true.